

# Stationary phase corrections in the process of bosonization of multi-quark interactions

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**Abstract.** The functional integration over the auxiliary bosonic variables of cubic order related with the effective action of the Nambu–Jona–Lasinio model with 't Hooft term has recently been obtained in the form of a loop expansion. Even numbers of loops contribute to the action, while odd numbers of loops are assigned to the measure. We consider the two-loop corrections and analyse their effect on the low-lying pseudoscalar and scalar mass spectra, quark condensates and weak decay constants. The results are compared to the leading order calculations and other approaches.

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## 1 Introduction

The large distance dynamics of QCD is dictated to a great extent by the spontaneous symmetry breaking of chiral symmetry [1, 2]. The Nambu–Jona–Lasinio (NJL) model of fermionic fields [3] suggests that the dynamical mechanism for such breaking is in analogy with the Ginsburg–Landau theory of superconductivity [4]. Numerous studies [5–7] have been performed since that time with the respective effective mesonic action derived from four-quark interactions of the NJL type. During these years the resolution of the  $U_A(1)$  problem has been found and, in particular, the relevance of the  $U(3)_L \times U(3)_R$  chiral symmetric NJL model combined with the six-quark 't Hooft flavour determinantal interaction (NJLH) [8] for low-energy phenomenology of mesons was noted [9–12]. The explicit breaking of the unwanted  $U_A(1)$  axial symmetry by the 't Hooft determinant is motivated by the instanton approach to low-energy QCD [8, 13].

Originally written in terms of fermionic degrees of freedom, the NJLH model has been widely explored at mean field level with Bethe–Salpeter and Hartree–Fock techniques applied to quark–antiquark scattering in its various channels of interaction [9, 14, 15].

In parallel, functional integral methods have been used to obtain the Lagrangian in bosonized form [10, 16–18]. The bosonization gives rise to a doubling of the mesonic

auxiliary fields, of which one set has to be integrated out. This latter, in the presence of the 't Hooft interaction, involves a term of cubic order, which cannot be integrated out exactly. In [10] the leading order stationary phase approximation (SPA) was calculated. At this order the effective potentials obtained with both methods coincide [17].

Given its success in describing a large bulk of empirical data, the question arises of whether corrections to the leading order SPA result are small. By embarking on this task, we came across a series of startling facts in our investigations [18]:

- (i) The stationary phase equations which one obtains in the NJLH model have more than one root (critical point). Only one has a regular behaviour in the limit  $\kappa \rightarrow 0$  of the six-quark coupling, the others are singular. The rigorous SPA treatment requires taking into account all critical points, which give rise to an unstable vacuum for the theory.
- (ii) The result obtained in [10] corresponds to the regular root contribution. It is an approximation which leads to an effective potential with a well-separated local minimum, which approaches smoothly the stable NJL vacuum as  $\kappa \rightarrow 0$ . Such a local minimum is probably a good ground for phenomenological estimates; at least all known calculations made in the NJLH model are based on this approximation. It has been shown recently [19] that eight-quark interactions stabilize this vacuum state, opening the way to justify this approach theoretically.
- (iii) Two expansions of the effective action have been considered: the perturbative series in  $\kappa$  and the loop expansion. Both of them were never studied beyond the

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leading order. It is tacitly assumed that next to the leading order corrections are small, although this fact has never been proven.

In this paper we quantify the two-loop order contributions to the Lagrangian derived previously by studying their impact on the mass spectrum of low-lying mesons. We show that the effect is of the order of a few percent compared to the leading order masses, improving them slightly. We think that it is rather safe to conclude that the loop expansion is rapidly converging, at least for the mass spectra.

The paper is structured as follows. In Sect. 2 we collect the essential information needed to extract the linear and quadratic terms, which contribute to the gap equations and mass terms, respectively. In Sect. 3 we write out the expressions for the gap equations, masses, weak decay constants and condensates. Section 4 contains the numerical results and discussion. Conclusions are presented in Sect. 5.

## 2 The ‘tandem’ Lagrangian

To be self contained and define the notation we review the main ingredients of our model calculations. The underlying multi-quark Lagrangian is bosonized in a two-step (tandem) process, in which a semi-bosonized functional, quadratic in the fermionic fields, and another functional, depending only on the auxiliary bosonic variables, can be dealt with separately. The integration over the quadratic fermionic degrees of freedom is formally exact and is calculated using a generalized heat kernel method. We start by presenting how these two types of functionals emerge and how we obtain and calculate the loop expansion we are after.

### 2.1 The stationary phase contribution

We consider the fermionic Lagrangian

$$\mathcal{L}_{\text{NJLH}} = \bar{q}(i\gamma^\mu\partial_\mu - m)q + \mathcal{L}_{\text{NJL}} + \mathcal{L}_{\text{H}}, \quad (1)$$

which contains the NJL four-quark vertices of the scalar and pseudoscalar types

$$\mathcal{L}_{\text{NJL}} = \frac{G}{2} [(\bar{q}\lambda_a q)^2 + (\bar{q}i\gamma_5\lambda_a q)^2] \quad (2)$$

and the six-quark ‘t Hooft interaction [8]

$$\mathcal{L}_{\text{H}} = \kappa(\det \bar{q}P_L q + \det \bar{q}P_R q), \quad (3)$$

where  $m$  is the diagonal current quark matrix for quark fields with  $N_f = 3$  flavours and  $N_c = 3$  colours. In (2)  $\lambda_a$ ,  $a = 0, 1, \dots, 8$ , are the normalized ( $\text{tr}\lambda_a\lambda_b = 2\delta_{ab}$ ) matrices in flavour space. The explicit form of these  $U(3)$  Hermitian generators is  $\lambda_0 = \sqrt{2/3}$ , and  $\lambda_a$  for  $a \neq 0$  are the usual Gell-Mann matrices. The positive coupling  $G$ ,  $[G] = \text{GeV}^{-2}$ , has order  $G \sim 1/N_c$ . In (3) the negative coupling  $\kappa$  of dimension  $[\kappa] = \text{GeV}^{-5}$  has the large- $N_c$  asymptotics

$\kappa \sim 1/N_c^{N_f}$ . Therefore,  $\mathcal{L}_{\text{NJLH}}$  dominates over  $\mathcal{L}_{\text{H}}$  at large  $N_c$ . The matrices  $P_{L,R} = (1 \mp \gamma_5)/2$  are projectors on the chiral states and the determinant is over flavour indices.

We are assuming that the quark fields transform like the fundamental representations of the global  $U(3)_L \times U(3)_R$  chiral group, i.e.

$$\delta q = i(\alpha + \gamma_5\beta)q, \quad \delta \bar{q} = -i\bar{q}(\alpha - \gamma_5\beta), \quad (4)$$

where the parameters of the infinitesimal transformations are chosen as  $\alpha = \alpha_a\lambda_a$ ,  $\beta = \beta_a\lambda_a$ . One now observes that

$$\begin{aligned} \delta \mathcal{L}_{\text{NJLH}} &= i\bar{q}([\alpha, m] - \gamma_5\{\beta, m\})q \\ &\quad + 2i\sqrt{6}\beta_0\kappa(\det \bar{q}P_R q - \det \bar{q}P_L q). \end{aligned} \quad (5)$$

The global chiral symmetry is broken explicitly by the current quark mass term and the  $U(1)_A$  axial symmetry is broken too due to the ‘t Hooft interaction.

The functional integral in bosonized form is derived in [10], and has the form

$$\begin{aligned} Z &= \int \prod_A \mathcal{D}\Pi_A \mathcal{D}q \mathcal{D}\bar{q} \exp\left(i \int d^4x \mathcal{L}_q(\bar{q}, q, \sigma, \phi)\right) \\ &\quad \times \int_{-\infty}^{+\infty} \prod_A \mathcal{D}R_A \exp\left(i \int d^4x \mathcal{L}_r(\Pi, \Delta; R)\right), \end{aligned} \quad (6)$$

where

$$\begin{aligned} \mathcal{L}_q &= \bar{q}(i\gamma^\mu\partial_\mu - M - \sigma - i\gamma_5\phi)q, \quad (7) \\ \mathcal{L}_r &= R_A(\Pi_A + \Delta_A) + \frac{G}{2} R_A^2 + \frac{\kappa}{3!} \Phi_{ABC} R_A R_B R_C, \end{aligned} \quad (8)$$

with a cubic polynomial in the fields  $R_A$  in the exponent. The notation is as follows [18]:  $R_A = (R_a, R_{\dot{a}}) = (s_a, p_a)$  and  $\Pi_A = (\Pi_a, \Pi_{\dot{a}}) = (\sigma_a, \phi_a)$  are a very compact way to represent two sets of auxiliary bosonic variables, each containing a scalar  $s_a$  ( $\sigma_a$ ) and a pseudoscalar  $p_a$  ( $\phi_a$ ) nonet. The indices  $(a, \dot{a})$  run from 0 to 8 independently in flavour space. We also define the related quantity  $\Delta_A = (\Delta_a, 0) = (M_a - m_a, 0)$ .

The external scalar fields  $\sigma = \sigma_a\lambda_a$  have been shifted  $\sigma_a \rightarrow \sigma_a + M_a$  by the constituent quark mass  $M_a$ , so that the expectation value of the shifted fields in the vacuum corresponding to dynamically broken chiral symmetry vanish. The vacuum expectation value of the ‘unshifted’ scalar field

$$\langle \sigma \rangle = M_a\lambda_a = \text{diag}(M_u, M_d, M_s) \quad (9)$$

gives the point where the effective potential of the model  $V(\langle \sigma \rangle)$  achieves its local minimum. The corresponding condition is known as the ‘gap’ equation. It eliminates tadpole graphs and determines the values of constituent quark masses as functions of the model parameters and of the cutoff  $\Lambda$ . The case  $m_u \neq m_d \neq m_s$  corresponds to the most general breakdown of the  $SU(3)$  flavour symmetry, giving  $M_u \neq M_d \neq M_s$ . In this way the ground state of the system includes effects of the explicit symmetry breaking. We will assume in the following that  $m_u = m_d$ .

The variables  $\sigma_a$  and  $\phi_a$  must be replaced by the physical scalar and pseudoscalar states  $\sigma_a^{ph}$  and  $\phi_a^{ph}$  determined through the appropriate normalization of their kinetic terms. Note that these terms, as well as other important contributions to the meson masses and interactions of the effective mesonic Lagrangian, are obtained as a result of integration over the quark fields in  $Z$ . One has

$$\sigma_a, \phi_a = g\sigma_a^{ph}, g\phi_a^{ph} \quad (10)$$

at leading order of the heat kernel expansion of the effective mesonic action (see Sect. 3). The quark–meson coupling  $g$ , being a function of parameters of the model, fulfils in addition the Goldberger–Treiman relation at the quark level:  $g = M_u/f_\pi$ . Combining this relation with (9) and (10) one finds the well-known linear sigma model result [20]

$$\langle \sigma_u^{ph} \rangle = f_\pi. \quad (11)$$

Finally, the three index coefficients  $\Phi_{ABC}$  are defined as

$$\Phi_{abc} = -\Phi_{ab\bar{c}} = \frac{3}{16}A_{abc}, \quad \Phi_{abc} = \Phi_{\bar{a}\bar{b}\bar{c}} = 0, \quad (12)$$

obeying

$$\Phi_{ABC}\delta_{BC} = 0. \quad (13)$$

The totally symmetric constants  $A_{abc}$  are related to the flavour determinant, and equal to

$$A_{abc} = \frac{1}{3!} \epsilon_{ijk} \epsilon_{mnl} (\lambda_a)_{im} (\lambda_b)_{jn} (\lambda_c)_{kl}. \quad (14)$$

Now the functional integral over the auxiliary variables  $R_A$  in (6),

$$\mathcal{Z}[\Pi, \Delta] \equiv \int_{-\infty}^{+\infty} \prod_A \mathcal{D}R_A \exp \left( i \int d^4x \mathcal{L}_r(\Pi, \Delta; R) \right), \quad (15)$$

can be written in the form [18]

$$\begin{aligned} \mathcal{Z}[\Pi, \Delta] &\sim \exp \left( i \int d^4x \mathcal{L}_{st} \right) \\ &\times \int_{-\infty}^{+\infty} \prod_A \mathcal{D}\bar{R}_A \exp \left( \frac{i}{2} \int d^4x \mathcal{L}''_{AB} \bar{R}_A \bar{R}_B \right) \\ &\times \sum_{n=0}^{\infty} \frac{1}{n!} \left( i \frac{\kappa}{3!} \Phi_{ABC} \int d^4x \bar{R}_A \bar{R}_B \bar{R}_C \right)^n. \end{aligned} \quad (16)$$

Here  $\mathcal{L}_{st}$  is the stationary value of the Lagrangian  $\mathcal{L}_r(\Pi, \Delta; R)$  associated with the regular critical point, around which the effective Lagrangian has been expanded. The barred fields indicate that they are shifted with respect to their stationary values  $\bar{R} = R - R_{st}$ . We denote by  $\mathcal{L}''_{AB}$  the coefficient of the expansion to second order in the fields. The expansion stops exactly at the third order

$\mathcal{L}'''_{ABC} = \kappa \Phi_{ABC}/3!$ , whose exponent is represented here as an infinite series.

The terms with odd values of  $n$  do not contribute in (16), because the corresponding functional integrals over  $\bar{R}_A$  are equal to zero. The first term,  $n = 0$ , sums all tree diagrams of a perturbative series in powers of the coupling  $\kappa$  resulting in  $\mathcal{L}_{st}$ . The  $n = 2$  term represents the first non-leading correction to the effective Lagrangian (for details we refer the reader to [18], where we show that this correction can be associated with a ‘two-loop’ contribution of quantum auxiliary bosonic fields)

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{st} + \left( \frac{\lambda}{2\pi} \right)^8 \frac{3\kappa^2 \mathcal{M}}{32N(N+2)(N+4)}. \quad (17)$$

The stationary Lagrangian reads to cubic order in the fields [16, 17]

$$\begin{aligned} \mathcal{L}_{st} &= h_a \sigma_a + \frac{1}{2} h_{ab}^{(1)} \sigma_a \sigma_b + \frac{1}{2} h_{ab}^{(2)} \phi_a \phi_b \\ &+ \frac{1}{3} \sigma_a \left[ h_{abc}^{(1)} \sigma_b \sigma_c + \left( h_{abc}^{(2)} + h_{bca}^{(3)} \right) \phi_b \phi_c \right] \\ &+ \mathcal{O}(\text{field}^4). \end{aligned} \quad (18)$$

The two-loop corrections, contained in the second term of (17), consist of a rather intricate dependence

$$\mathcal{M} = \left( \text{tr} \mathcal{L}''^{-1} \right)^3 + 6 \text{tr} \mathcal{L}''^{-1} \text{tr} \left( \mathcal{L}''^{-1} \right)^2 + 8 \text{tr} \left( \mathcal{L}''^{-1} \right)^3 \quad (19)$$

on flavour traces of powers of  $\mathcal{L}''^{-1}$ , which is the inverse of the real and symmetric  $N \times N$  matrix ( $N = 18$ )

$$\mathcal{L}''_{AB}(R_{st}) = \begin{pmatrix} G\delta_{ab} + \frac{3\kappa}{16} A_{abc} s_{st}^c & -\frac{3\kappa}{16} A_{abc} p_{st}^c \\ -\frac{3\kappa}{16} A_{abc} p_{st}^c & G\delta_{ab} - \frac{3\kappa}{16} A_{abc} s_{st}^c \end{pmatrix}, \quad (20)$$

calculated at the stationary points  $s_{st}^a$  and  $p_{st}^a$ . These are expressed in increasing powers of the external fields  $\sigma_a, \phi_a$  as

$$\begin{aligned} s_{st}^a &= h_a + h_{ab}^{(1)} \sigma_b + h_{abc}^{(1)} \sigma_b \sigma_c + h_{abc}^{(2)} \phi_b \phi_c \\ &+ h_{abcd}^{(1)} \sigma_b \sigma_c \sigma_d + h_{abcd}^{(2)} \sigma_b \phi_c \phi_d + \dots, \end{aligned} \quad (21)$$

$$\begin{aligned} p_{st}^a &= h_{ab}^{(2)} \phi_b + h_{abc}^{(3)} \phi_b \sigma_c + h_{abcd}^{(3)} \sigma_b \sigma_c \phi_d \\ &+ h_{abcd}^{(4)} \phi_b \phi_c \phi_d + \dots, \end{aligned} \quad (22)$$

with  $h_{ab\dots}^{(i)}$  depending on  $\Delta_a$  and coupling constants (see the appendix). In particular, the coefficients have non-vanishing components for  $a = (0, 3, 8)$  and are obtained, with  $h = h_a \lambda_a = \text{diag}(h_u, h_d, h_s)$ , in the case of isotopic symmetry ( $h_u = h_d$ ) as [17]

$$\begin{cases} Gh_u + \Delta_u + \frac{\kappa}{16} h_u h_s = 0, \\ Gh_s + \Delta_s + \frac{\kappa}{16} h_u^2 = 0. \end{cases} \quad (23)$$

In (17)  $\lambda$  denotes an ultraviolet cutoff associated with the stationary phase corrections to the functional integral over auxiliary bosonic fields. It is a free parameter to be fixed by phenomenology.

To handle the new contribution  $\mathcal{M}$ , we expand  $\mathcal{L}''_{AB}(R_{\text{st}})$ , which we abbreviate from now on as  $\mathcal{L}''$ , to second order in the external fields  $\sigma_a, \phi_a$  as

$$\mathcal{L}'' = L_0 + L_1 + L_2 + \mathcal{O}(\text{field}^3), \quad (24)$$

and its inverse

$$\mathcal{L}''^{-1} = \bar{L}_0 + \bar{L}_1 + \bar{L}_2 + \mathcal{O}(\text{field}^3). \quad (25)$$

This is all one needs to extract the relevant terms for the masses arising from the two-loop correction term. Here  $L_i$ ,  $i = 0, 1, 2$ , denote the matrices which are constant, linear and quadratic in the fields, respectively. The  $\bar{L}_i$  are constructed order by order, starting from the 0th order

$$\mathcal{L}'' \mathcal{L}''^{-1} = L_0 \bar{L}_0 = 1, \quad (26)$$

i.e.  $\bar{L}_0 = L_0^{-1}$ .

The next terms are conditioned by this relation. Combining the first-order Lagrangians and truncating at the linear fields

$$(L_0 + L_1)(\bar{L}_0 + \bar{L}_1) \rightarrow L_0 \bar{L}_0 + L_1 \bar{L}_0 + L_0 \bar{L}_1 = 1, \quad (27)$$

one obtains the matrix  $\bar{L}_1$ , after using (26),

$$\bar{L}_1 = -\bar{L}_0 L_1 \bar{L}_0. \quad (28)$$

In a similar fashion one derives the matrix  $\bar{L}_2$  as

$$\begin{aligned} \bar{L}_2 &= -(\bar{L}_0 L_2 \bar{L}_0 + \bar{L}_0 L_1 \bar{L}_1) \\ &= -(\bar{L}_0 L_2 \bar{L}_0 - \bar{L}_0 L_1 \bar{L}_0 L_1 \bar{L}_0). \end{aligned} \quad (29)$$

Using  $\bar{L}_i$  in (25) and inserting in (19), one obtains

$$\begin{aligned} \mathcal{M}_2 &= 3 \left\{ (\text{tr} \bar{L}_0) (\text{tr} \bar{L}_1)^2 + (\text{tr} \bar{L}_0)^2 (\text{tr} \bar{L}_2) \right. \\ &\quad \left. + 8 \text{tr} (\bar{L}_0 \bar{L}_1^2 + \bar{L}_2 \bar{L}_0^2) \right\} \\ &\quad + 6 \left\{ \text{tr} (\bar{L}_0) [\text{tr} (\bar{L}_1^2) + 2 \text{tr} (\bar{L}_0 \bar{L}_2)] \right. \\ &\quad \left. + 2 \text{tr} (\bar{L}_1) \text{tr} (\bar{L}_0 \bar{L}_1) + (\text{tr} \bar{L}_2) \text{tr} (\bar{L}_0^2) \right\}, \end{aligned} \quad (30)$$

where  $\mathcal{M}_2$  stands for the part of  $\mathcal{M}$  which contains only the second-order terms in the fields  $\sigma_a, \phi_a$ .

This expression is used in ‘Mathematica’ [21]. Although the results, after evaluation of traces, are analytical they are very lengthy and not illuminating, and will not be presented here. However, some structures are relevant for the low-energy theorems and the results will be encrypted in them, as shown in Sect. 3.

## 2.2 The heat kernel contribution

It still remains to evaluate the functional integral over the quark degrees of freedom in (6). The Lagrangian  $\mathcal{L}_q$  is invariant under the chiral transformations (4) and the transformations

$$\delta\sigma = i[\alpha, \sigma + M] + \{\beta, \phi\}, \quad \delta\phi = i[\alpha, \phi] - \{\beta, \sigma + M\}, \quad (31)$$

induced by them for the external fields. All symmetry breaking terms have been absorbed in  $\mathcal{L}_r$ . This fact is of importance, since one can use then the generalized asymptotic expansion of the quark determinant [22, 23]. This method preserves the above-mentioned symmetry at any order, taking into account the effects of the flavour symmetry breaking contained in the mass matrix  $M$ . Thus, the corresponding part of the effective action can be written as

$$\ln |\det D| = -\frac{1}{32\pi^2} \int d^4x_E \sum_{i=0}^{\infty} I_{i-1} \text{tr}(b_i), \quad (32)$$

where  $D = i\gamma_\mu \partial_\mu - M - \sigma - i\gamma_5 \phi$  is the Dirac operator present in  $\mathcal{L}_q$  (7), and the  $b_i$  are generalized Seeley–DeWitt coefficients [22], of which we show the first four for the case of  $SU(2)_I \times U(1)_Y$  flavour symmetry:

$$\begin{aligned} b_0 &= 1, \\ b_1 &= -Y, \\ b_2 &= \frac{Y^2}{2} + \frac{\Delta_{us}}{\sqrt{3}} \lambda_8 Y, \\ b_3 &= -\frac{Y^3}{3!} + \frac{\Delta_{us}^2}{6\sqrt{3}} \lambda_8 Y - \frac{\Delta_{us}}{2\sqrt{3}} \lambda_8 Y^2 - \frac{1}{12} (\partial Y)^2. \end{aligned} \quad (33)$$

In the present case the background-dependent structure  $Y$  is given by

$$Y = i\gamma_\mu (\partial_\mu \sigma + i\gamma_5 \partial_\mu \phi) + \sigma^2 + \{M, \sigma\} + \phi^2 + i\gamma_5 [\sigma + M, \phi]. \quad (34)$$

We use the definition  $\Delta_{ij} \equiv M_i^2 - M_j^2$ . In (32) the trace is to be taken over colour, flavour and Dirac 4-spinor indices and the regulator-dependent integrals  $I_i$  are the weighted sums

$$I_i = \frac{1}{3} (2J_i(M_u^2) + J_i(M_s^2)) \quad (35)$$

with

$$J_i(M_j^2) = \int_0^\infty \frac{dt}{t^{2-i}} \rho(t\Lambda^2) \exp(-tM_j^2). \quad (36)$$

They are regularized with the Pauli–Villars regularization scheme [24] with two subtractions and one ultraviolet cutoff  $\Lambda$

$$\rho(t\Lambda^2) = 1 - (1 + t\Lambda^2) \exp(-t\Lambda^2). \quad (37)$$

We obtain, for instance [25]

$$J_0(M^2) = \Lambda^2 - M^2 \ln \left( 1 + \frac{\Lambda^2}{M^2} \right), \quad (38)$$

$$J_1(M^2) = \ln \left( 1 + \frac{\Lambda^2}{M^2} \right) - \frac{\Lambda^2}{\Lambda^2 + M^2}. \quad (39)$$

Both of them are divergent in the limiting case  $\Lambda \rightarrow \infty$ . Note that  $\Lambda$  does not need to be the same cutoff as  $\lambda$  of (17). In the following we restrict our study to the two non-trivial terms,  $b_1$  and  $b_2$ , in the asymptotic expansion of  $\ln |\det D|$ . In this case only  $I_0$  and  $I_1$  are involved, related to the quark one-loop integrals of one- and two-point functions, respectively, at zero four-momentum transfer.

### 3 Gap equations, condensates and meson spectra

#### 3.1 Gap equations and condensates

The complete effective bosonized Lagrangian

$$\mathcal{L}_b = \mathcal{L}_{\text{HK}} + \mathcal{L}_{\text{st}} + \mathcal{L}_c \quad (40)$$

comprises contributions from the heat kernel expansion to order  $b_2$ ,  $\mathcal{L}_{\text{HK}}$ , and from (17), where  $\mathcal{L}_c$  stands for the two-loop corrections. We restrict to the case of  $SU(2)_I \times U(1)_Y$  symmetry, e.g.  $M_u = M_d \neq M_s$ . The first two contributions remain the same as in the leading order calculations [12].

Equating the coefficient of  $\sigma_i$ ,  $i = (u, d, s)$ , in (40) to zero we obtain the gap equations

$$\begin{aligned} h_u + \frac{N_c}{6\pi^2} M_u [3I_0 + (M_s^2 - M_u^2)I_1] + 2c_u &= 0, \\ h_s + \frac{N_c}{6\pi^2} M_s [3I_0 - 2(M_s^2 - M_u^2)I_1] + 2c_s &= 0, \end{aligned} \quad (41)$$

where  $c_u$  and  $c_s$  denote the corrections arising from  $\mathcal{L}_c$ . They depend on  $h_u, h_s, \lambda, \kappa$ . These equations must be solved self-consistently and in conjunction with the stationary phase conditions (23). The solutions  $M_i$  of (41) allow us to calculate the condensates  $\langle \bar{u}u \rangle$  and  $\langle \bar{s}s \rangle$  (see (54))

$$\langle \bar{q}_i q_i \rangle = -\frac{N_c}{4\pi^2} [M_i J_0(M_i^2) - m_i J_0(m_i^2)], \quad (42)$$

where we have subtracted the contribution from the trivial vacuum [9]. Although they are structurally identical to the condensates calculated at leading order, they encode the information of the correction terms  $c_i$  implicitly through  $M_i$ .

#### 3.2 Meson masses

The expressions for the leading order masses, i.e. with  $\mathcal{L}_c$  put to zero, will not be repeated here. They were obtained in [12]. The correction mass terms can just be added to the leading order terms in their ‘raw’ form, that is, as they

are directly extracted from  $\mathcal{L}_{\text{HK}}$ , depending on  $I_0, I_1$  integrals. To check the low-energy theorems, one can then use the new gap equations, with the correction terms  $c_i$  included, to eliminate these integrals. For example, for the pion,  $\phi_j$  ( $j = 1, 2, 3$ ), one has

$$\mathcal{L}_{\text{HK}}(m_\pi^2) = \frac{N_c}{12\pi^2} (3I_0 + \Delta_{su} I_1) \phi_j^2, \quad (43)$$

$$\mathcal{L}_{\text{st}}(m_\pi^2) = -\frac{\phi_j^2}{2G(1 + \omega_s)}. \quad (44)$$

With ‘Mathematica’ we are able to identify

$$\mathcal{L}_c(m_\pi^2) = -\frac{c_u \phi_j^2}{(4G)^2 \omega_u (1 + \omega_s)}, \quad (45)$$

where  $\omega_i = \kappa h_i / (16G)$ . This connection between the pion mass correction and the gap equation correction term  $c_u$  is crucial to guarantee the Goldstone limit. Indeed we obtain, after eliminating  $I_0, I_1$  from (43) with help of the gap equations,

$$\begin{aligned} m_\pi^2 &= \overset{\circ}{m}_\pi^2 \left( 1 + 2 \frac{c_u}{h_u} \right), \\ \overset{\circ}{m}_\pi^2 &= \frac{g^2 m_u}{M_u G (1 + \omega_s)}, \end{aligned} \quad (46)$$

where  $\overset{\circ}{m}_\pi$  is structurally identical with the leading order pion mass;  $g^2 = 4\pi^2 / (N_c I_1)$  renormalizes the pion fields to the physical fields (see (10)). We also used

$$-\frac{h_u}{\Delta_u} = \frac{1}{G(1 + \omega_s)}, \quad (47)$$

which is a simple consequence of the stationary phase conditions (23).

In an analogous way we are able to find for the kaon mass

$$\begin{aligned} m_K^2 &= \overset{\circ}{m}_K^2 \left( 1 + 2 \frac{c_u + c_s}{h_u + h_s} \right), \\ \overset{\circ}{m}_K^2 &= \frac{g^2 (m_u + m_s)}{G(M_u + M_s)(1 + \omega_u)}, \end{aligned} \quad (48)$$

where  $\overset{\circ}{m}_K$  has the form of the leading order kaon mass. We again used (23) to obtain

$$-\frac{h_u + h_s}{\Delta_u + \Delta_s} = \frac{1}{G(1 + \omega_u)}. \quad (49)$$

Concerning the  $\eta, \eta'$  corrections, we show here only some relevant properties obtained with the help of ‘Mathematica’ in the  $SU(3)$  limit:

$$\begin{aligned} (\Delta m_\pi^2)_c &= (\Delta m_K^2)_c = (\Delta m_{88}^2)_c, \\ (\Delta m_{08}^2)_c &= 0, \\ (\Delta m_{00}^2)_c - (\Delta m_{88}^2)_c &\neq 0, \end{aligned} \quad (50)$$

where, for instance,  $(\Delta m_\pi^2)_c$  is the contribution to the pion mass obtained from the Lagrangian  $\mathcal{L}_c$ . Therefore, the corrections to the flavour (0, 8) components follow the same

**Table 1.** The main parameters of the model: current quark masses,  $m_u$ ,  $m_s$ , and corresponding constituent masses,  $M_u$ ,  $M_s$ , in MeV, couplings  $G$  (in  $\text{GeV}^{-2}$ ) and  $\kappa$  (in  $\text{GeV}^{-5}$ ) and two cutoffs  $\Lambda$ ,  $\lambda$  in GeV. The values of condensates are given in MeV

	$m_u$	$m_s$	$M_u$	$M_s$	$G$	$-\kappa$	$\Lambda$	$\lambda$	$-\langle\bar{u}u\rangle^{1/3}$	$-\langle\bar{s}s\rangle^{1/3}$
a	6.3	194	398	588	13.5	1300*	0.82	0*	229	172
b	6.3	194	398	588	13.5	1300*	0.82	1.8*	229	172
c	6.3	194	398	588	13.4	1370*	0.82	0*	229	172
d	6.3	194	398	588	11.8	1370*	0.82	1.7*	229	172
e	2.8	92	216	385	3.14	120	1.37	0*	302	314
f	2.1	69	196	354	2.15	53	1.64	1.9*	333	363

patterns as in the leading order case [12], complying with the low-energy requirements: the octet member  $m_{88}^2$  remains degenerate with the pion and kaon, the mixing  $m_{08}^2$  vanishes and in the chiral limit the correction to the mass of the singlet  $m_{00}^2$  is also non-vanishing, and will therefore contribute to the singlet–octet splitting.

For the scalars we obtain in the SU(3) limit also that the corrections to the masses behave as

$$\begin{aligned} (\Delta M_{a_0}^2)_c &= (\Delta M_{K^*}^2)_c = (\Delta M_{88}^2)_c, \\ (\Delta M_{08}^2)_c &= 0, \\ (\Delta M_{00}^2)_c - (\Delta M_{88}^2)_c &\neq 0. \end{aligned} \quad (51)$$

### 3.3 Weak decay constants

We use the partially conserved axial current condition (PCAC) and the Gell-Mann–Oakes–Renner (GOR) [26] relation to extract the condensates. From PCAC the weak decay constants are given as

$$f_\pi = \frac{M_u}{g}, \quad f_K = \frac{M_u + M_s}{2g}. \quad (52)$$

Using this and the mass relations for  $m_\pi$  and  $m_K$ , (46) and (48), one obtains the GOR equations with some model corrections of higher order in the current quark masses and from which one identifies the condensates

$$\begin{aligned} m_\pi^2 f_\pi^2 &= m_u (h_u + 2c_u) \left(1 + \frac{m_u}{\Delta_u}\right) \\ &= -2m_u \langle 0 | \bar{u}u | 0 \rangle \left(1 + \frac{m_u}{\Delta_u}\right), \\ m_K^2 f_K^2 &= \frac{1}{4} (m_u + m_s) (h_u + 2c_u + h_s + 2c_s) \left(1 + \frac{m_u + m_s}{\Delta_u + \Delta_s}\right) \\ &= -\frac{1}{2} (m_u + m_s) \langle 0 | \bar{u}u + \bar{s}s | 0 \rangle \left(1 + \frac{m_u + m_s}{\Delta_u + \Delta_s}\right). \end{aligned} \quad (53)$$

Finally, by using the gap equations (41) in (53) and expressing  $I_0$ ,  $I_1$  through  $J_0(M_i^2)$ ,  $J_1(M_i^2)$  with (35), we obtain the condensates as

$$\langle 0 | \bar{q}_i q_i | 0 \rangle = -\frac{N_c}{4\pi^2} M_i J_0(M_i^2) + \mathcal{O}(J_2), \quad (54)$$

where the  $\mathcal{O}(J_2)$  terms are neglected, to conform with the truncation of the heat kernel series. We recall that the  $\mathcal{O}(J_2)$  emerge from a property of the generalized heat kernel series in which differences of  $J_k(M_u^2) - J_k(M_s^2)$  are expressed as an infinite series involving  $J_{k+l}$ ,  $l > 0$  [22].

## 4 Numerical results and discussion

There are six parameters in the model,  $m_u$ ,  $m_s$ ,  $G$ ,  $\kappa$ ,  $\Lambda$ ,  $\lambda$ . To see the effects of the new contribution, proportional to the cutoff  $\lambda$ , we compare pairwise in the sets (a, b), (c, d) and (e, f) of Tables 1–3 the results calculated with  $\lambda = 0$  and  $\lambda \neq 0$ , keeping the remaining input unchanged. In this way, within each pair of sets, a running value of  $\lambda$  between the indicated ones will interpolate smoothly between the calculated observables shown. In a–d we fix four parameters through the pseudoscalar sector,  $m_\pi$ ,  $m_K$ ,  $f_\pi$ ,  $f_K$ , and adjust  $\kappa$  through the quark condensate  $\langle\bar{u}u\rangle$ . In sets (e, f) five parameters are fixed through  $m_\pi$ ,  $m_K$ ,  $f_\pi$ ,  $m_{\eta'}$ , and the scalar  $a_0$ . Input is indicated through a \*.

It is clear that the large differences observed among different pairs of sets come from the leading order contribution. For example, the condensates,  $f_K$  and the  $\eta$  mass are strongly dependent on the value of  $\kappa$ , which is one order of magnitude larger in the sets (a–d) as compared to sets (e, f). This observation applies also to the scalar spectrum, with large changes resulting at leading order. They are best described in set (f), but this implies rather large values for  $f_K$  and the condensates.

**Table 2.** The main characteristics of the light pseudoscalar mesons in MeV. The singlet–octet mixing angle  $\theta_p$  is given in degrees

	$m_\pi$	$m_K$	$f_\pi$	$f_K$	$m_\eta$	$m_{\eta'}$	$\theta_p$
a	138*	494*	92*	114*	476	986	−14
b	138*	494*	92*	114*	487	958	−15
c	138*	494*	92*	114*	480	1020	−13
d	138*	494*	92*	114*	472	959	−15
e	138*	494*	92*	129	533	1097*	−1.2
f	138*	494*	92*	129	540	1097*	0.5

**Table 3.** The characteristics of the light scalar nonet in MeV and the singlet–octet mixing angle  $\theta_s$  in degrees

	$m_{a_0} \sim a_0(980)$	$m_{K_0^*} \sim K_0^*(800)$	$m_\sigma \sim f_0(600)$	$m_{\sigma'} \sim f_0(980)$	$\theta_s$
a	1040	1267	806	1438	24
b	981	1219	781	1427	24
c	1056	1280	805	1447	23.7
d	967	1208	762	1426	23.5
e	980*	1029	413	1123	19.5
f	980*	992	346	1073	18

One observes however that the corrections, although small, have the correct trend, diminishing the splitting in the singlet–octet members of the pseudoscalar and scalar spectra. Comparing sets (a, b) with (c, d), one sees that by enlarging the magnitude of  $\kappa$ , the effect of the corrections becomes stronger, even for a smaller value of  $\lambda$ .

We also remark the interesting fact that  $\lambda$  cannot be arbitrarily increased, maintaining the remaining input on observables fixed. At values quite close to the ones indicated, solutions cease to exist. There is an intrinsic constraint on the size of corrections.

One might be struck by the large variation in the values of  $\kappa$  and their relation to the convergence of the loop expansion (16). This can be understood by identifying the dimensionless expansion parameter of the series. Following [18], where standard methods are used to justify the stationary phase approach to functional integrals, one obtains the dimensionless parameter

$$\zeta = \frac{\kappa^2}{32G^3} \left( \frac{\lambda}{2\pi} \right)^4. \quad (55)$$

Here the group structure factor  $3/16$  of  $\Phi_{abc}$  in (12) as well as the factor  $1/3!$  appearing at each order in (16) have been taken into account. Note also that each term of the expansion carries a further suppression factor  $1/n!$ . In the present case  $n = 2$ , so that one finds for sets (b), (d), (f) that  $\zeta^2/2$  is 0.0096, 0.021, 0.0023, respectively. This attests for a fast convergence of the series.

For a comparison with empirical values, we take from [27]:

$$\begin{aligned} m_u &= 1.5\text{--}4 \text{ MeV}, \\ m_d &= 4\text{--}8 \text{ MeV}, \\ m_s &= 80\text{--}130 \text{ MeV}, \\ m_{\pi^\pm} &= 139.57018 \pm 0.00035 \text{ MeV}, \\ m_{K^\pm} &= 493.677 \pm 0.016 \text{ MeV}, \\ m_\eta &= 547 \pm 0.12 \text{ MeV}, \\ m_{\eta'} &= 957.78 \pm 0.14 \text{ MeV}, \end{aligned} \quad (56)$$

for the masses in the low-lying pseudoscalar sector. The weak decay constants  $f_\pi^e = 130.7 \pm 0.1 \pm 0.36 \text{ MeV}$ ,  $f_K^e = 159.8 \pm 1.4 \pm 0.44 \text{ MeV}$  relate to ours through a  $\sqrt{2}$  normalization factor; thus  $f_\pi \simeq 92.4 \text{ MeV}$  and  $f_K \simeq 113 \text{ MeV}$ .

The low-lying scalar masses are presently<sup>1</sup>

$$\begin{aligned} m_{a_0(980)} &= 984.7 \pm 1.2 \text{ MeV}, \\ m_{f_0(600)} &= 400\text{--}1200 \text{ MeV}, \\ m_{f_0(980)} &= 980 \pm 10 \text{ MeV}, \\ m_{K_0^*(800)} &= 701\text{--}970 \text{ MeV}. \end{aligned} \quad (57)$$

The recent update of the light-quark condensate is  $\langle(\bar{u}u + \bar{d}d)/2\rangle(1 \text{ GeV}) = -(242 \pm 15 \text{ MeV})^3$ ; the flavour breaking ratio is known to be  $\langle\bar{s}s\rangle/\langle(\bar{u}u + \bar{d}d)/2\rangle = 0.8 \pm 0.3$  [28].

Finally, as our numerical calculations do not differ significantly from the leading order values, we refer the reader to [12] where a thorough discussion of our leading order results is made in comparison with the ones obtained from other approaches [29–35].

## 5 Conclusions

The bosonization of the model combining the NJL and the 't Hooft multi-quark interactions leads to corrections associated with the stationary phase integration over auxiliary bosonic variables in the functional integral of the theory. The purpose of the present work has been to quantify the next to leading order (NLO) corrections, and to study their phenomenological effect on the mass spectrum of light pseudoscalar and scalar mesons.

To this end, the first correction to the tree-level effective action has been considered. We have obtained the linear and quadratic terms (in the external mesonic fields) of the NLO Lagrangian. The group structure of the  $SU(2) \times U(1)$  flavour symmetry considered leads to quite intricate expressions for the mass corrections. We have shown in a transparent way that they comply with the QCD low-energy theorems. We have calculated the mass spectra of the low-lying pseudoscalars and scalars, quark condensates and weak decay constants  $f_\pi, f_K$ . The corrections are small and improve slightly the leading order results.

We conclude from these calculations that the series considered is well convergent. It is an important conclusion, because it justifies the leading order estimates made

<sup>1</sup> As several data sets are presented in [27] for  $m_{K_0^*(800)}$ , please consult it for details. Here we indicate the lowest and the highest values collected from all samples.

before, on one hand, and reports the self consistency of the stationary phase approach applied to the bosonization of effective multi-quark interactions, on the other hand.

At the same time one may still expect some noticeable effects of the NLO terms which can show themselves in third and higher order mesonic amplitudes, especially in the cases where there is a strong cancellation between the tree-level contributions.

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## Appendix

The equations and algebra leading to the coefficients  $h_{ab\dots}^{(i)}$  of the series for  $s_a$  and  $p_a$  can be found in [17]. The explicit expressions for the case of one and two lower indices are also given there. In this appendix we collect explicit expressions for the coefficients  $h_{abc}^{(1)}$  and  $h_{abc}^{(2)}$  entering in the expansion of  $s_a$  and needed for the evaluation of meson mass terms. Due to the trace structure of (30) and since  $\bar{L}_0$  contributes only in the block diagonal, with non-vanishing entries in the diagonal and elements of 0, 8 mixing, only the elements of the diagonal and (0, 8) mixing of  $L_2$  will contribute to the mass terms. For those we need

$$\begin{aligned} h_{0aa}^{(1)} &= \frac{\kappa}{16\sqrt{6}G^3} \frac{1+\omega_s-2\omega_u}{\mu_+(1-\omega_s)^2} \quad \text{for } a \in 1, 2, 3. \\ h_{0aa}^{(1)} &= \frac{\kappa}{16\sqrt{6}G^3} \frac{1}{\mu_+(1-\omega_u)} \quad \text{for } a \in 4, 5, 6, 7. \\ h_{8aa}^{(1)} &= -\frac{\kappa}{16\sqrt{3}G^3} \frac{1+\omega_s+\omega_u}{\mu_+(1-\omega_s)^2} \quad \text{for } a \in 1, 2, 3. \\ h_{8aa}^{(1)} &= \frac{\kappa}{32\sqrt{3}G^3} \frac{1+2\omega_u}{\mu_+(1-\omega_u)^2} \quad \text{for } a \in 4, 5, 6, 7. \end{aligned} \quad (\text{A.1})$$

$$\begin{aligned} h_{000}^{(1)} &= -\frac{\kappa}{8\sqrt{6}G^3\mu_+^3} (1+\omega_s-2\omega_u)(1-\omega_u)^2 \\ h_{088}^{(1)} &= \frac{\kappa}{16\sqrt{6}G^3\mu_+^3} (1+2\omega_u)[1+\omega_s(1-2\omega_u)] \\ h_{008}^{(1)} &= h_{080}^{(1)} = h_{800}^{(1)} = \frac{\kappa}{8\sqrt{3}G^3\mu_+^3} \omega_u(\omega_u-\omega_s)(1-\omega_u) \\ h_{808}^{(1)} &= h_{880}^{(1)} = \frac{\kappa}{16\sqrt{6}G^3\mu_+^3} (1+\omega_s+2\omega_u-4\omega_s\omega_u^2) \\ h_{888}^{(1)} &= \frac{\kappa}{16\sqrt{3}G^3\mu_+^3} (1+\omega_u+\omega_s)(1+2\omega_u)^2 \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} h_{0aa}^{(2)} &= -\frac{\kappa}{16\sqrt{6}G^3} \frac{1+\omega_s-2\omega_u}{\mu_+(1+\omega_s)^2} \quad \text{for } a \in 1, 2, 3. \\ h_{0aa}^{(2)} &= -\frac{\kappa}{16\sqrt{6}G^3} \frac{1-\omega_u}{\mu_+(1+\omega_u)^2} \quad \text{for } a \in 4, 5, 6, 7. \end{aligned}$$

$$\begin{aligned} h_{8aa}^{(2)} &= \frac{\kappa}{16\sqrt{3}G^3} \frac{1+\omega_s+\omega_u}{\mu_+(1+\omega_s)^2} \quad \text{for } a \in 1, 2, 3. \\ h_{8aa}^{(2)} &= -\frac{\kappa}{32\sqrt{3}G^3} \frac{1+2\omega_u}{\mu_+(1+\omega_u)^2} \quad \text{for } a \in 4, 5, 6, 7. \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} h_{000}^{(2)} &= \frac{\kappa}{24\sqrt{6}G^3\mu_+\mu_-^2} (1+\omega_u)(3+\omega_u-\omega_s+3\omega_u\omega_s-6\omega_u^2) \\ h_{088}^{(2)} &= -\frac{\kappa}{48\sqrt{6}G^3\mu_+\mu_-^2} (1-2\omega_u)[3-4\omega_u-\omega_s(5-6\omega_u)] \\ h_{008}^{(2)} &= h_{080}^{(2)} = -\frac{\kappa}{24\sqrt{3}G^3\mu_+\mu_-^2} (\omega_u-\omega_s)(1+\omega_u-3\omega_u^2) \\ h_{800}^{(2)} &= \frac{\kappa}{24\sqrt{3}G^3\mu_+\mu_-^2} (\omega_u-\omega_s)(1+\omega_u)(2+3\omega_u) \\ h_{808}^{(2)} &= h_{880}^{(2)} = -\frac{\kappa}{48\sqrt{6}G^3\mu_+\mu_-^2} \\ &\quad \times (3+2\omega_u-4\omega_u^2+\omega_s(1-8\omega_u-12\omega_u^2)) \\ h_{888}^{(2)} &= -\frac{\kappa}{48\sqrt{3}G^3\mu_+\mu_-^2} \\ &\quad \times (1-2\omega_u)(3+\omega_u-\omega_s-6\omega_u\omega_s-6\omega_u^2), \end{aligned} \quad (\text{A.4})$$

where

$$\omega_i = \frac{\kappa h_i}{16G}, \quad \mu_{\pm} = 1 \pm \omega_s - 2\omega_u^2. \quad (\text{A.5})$$

## References

1. S. Weinberg, *Physica A* **96**, 327 (1979)
2. J. Gasser, H. Leutwyler, *Ann. Phys. (N. Y.)* **158**, 142 (1984); J. Gasser, H. Leutwyler, *Nucl. Phys. B* **250**, 465 (1985)
3. Y. Nambu, G. Jona-Lasinio, *Phys. Rev.* **122**, 345 (1961); *Phys. Rev.* **124**, 246 (1961); V.G. Vaks, A.I. Larkin, *Zh. Eksp. Teor. Fiz.* **40**, 282 (1961)
4. V.L. Ginsburg, L.D. Landau, *Zh. Eksp. Teor. Fiz.* **20**, 1067 (1950)
5. T. Eguchi, H. Sugawara, *Phys. Rev. D* **10**, 4257 (1974); T. Eguchi, *Phys. Rev. D* **14**, 2755 (1976); K. Kikkawa, *Prog. Theor. Phys.* **56**, 947 (1976)
6. M.K. Volkov, D. Ebert, *Sov. J. Nucl. Phys.* **36**, 736 (1982); D. Ebert, M.K. Volkov, *Z. Phys. C* **16**, 205 (1983)
7. M.K. Volkov, *Ann. Phys. (N. Y.)* **157**, 282 (1984); A. Dhar, S. Wadia, *Phys. Rev. Lett.* **52**, 959 (1984); A. Dhar, R. Shankar, S. Wadia, *Phys. Rev. D* **31**, 3256 (1985); D. Ebert, H. Reinhardt, *Nucl. Phys. B* **271**, 188 (1986); C. Schüren, E.R. Arriola, K. Goetze, *Nucl. Phys. A* **547**, 612 (1992); J. Bijnens, C. Bruno, E. de Rafael, *Nucl. Phys. B* **390**, 501 (1993) [hep-ph/9206236]; V. Bernard, A.A. Osipov, U.-G. Meißner, *Phys. Lett. B* **324**, 201 (1994) [hep-ph/9312203]; V. Bernard, A.H. Blin, B. Hiller, Yu.P. Ivanov, A.A. Osipov, U.-G. Meißner, *Ann. Phys. (N. Y.)* **249**, 499 (1996) [hep-ph/9506309]; J. Bijnens, *Phys. Rep.* **265**, 369 (1996) [hep-ph/9502335]; A.A. Osipov, B. Hiller, *Phys. Rev. D* **62**, 114013 (2000) [hep-ph/0007102]; A.A. Osipov, M. Sampaio, B. Hiller, *Nucl. Phys. A* **703**, 378 (2002) [hep-ph/0110285]



8. G. 't Hooft, Phys. Rev. D **14**, 3432 (1976); Erratum, Phys. Rev. D **18**, 2199 (1978)
9. V. Bernard, R.L. Jaffe, U.-G. Meissner, Phys. Lett. B **198**, 92 (1987); V. Bernard, R.L. Jaffe, U.-G. Meissner, Nucl. Phys. B **308**, 753 (1988)
10. H. Reinhardt, R. Alkofer, Phys. Lett. B **207**, 482 (1988)
11. S.P. Klevansky, Rev. Mod. Phys. **64**, 649 (1992); T. Hatsuda, T. Kunihiro, Phys. Rep. **247**, 221 (1994) [hep-ph/9401310]
12. A.A. Osipov, H. Hansen, B. Hiller, Nucl. Phys. A **745**, 81 (2004) [hep-ph/0406112]
13. D. Diakonov, V. Petrov, Sov. Phys. JETP **62**, 204, 431 (1985); Nucl. Phys. B **272**, 457 (1986); D. Diakonov, V. Petrov, Leningr. Prepr. 1153 (1986); D. Diakonov, V. Petrov, P. Pobylytsa, Nucl. Phys. B **306**, 809 (1988)
14. S. Klimt, M. Lutz, U. Vogl, W. Weise, Nucl. Phys. A **516**, 429 (1990); U. Vogl, M. Lutz, S. Klimt, W. Weise, Nucl. Phys. A **516**, 469 (1990)
15. V. Bernard, A.H. Blin, B. Hiller, U.-G. Meißner, M.C. Riuvo, Phys. Lett. B **305**, 163 (1993) [hep-ph/9302245]; V. Dmitrasinovic, Nucl. Phys. A **686**, 379 (2001) [hep-ph/0010047]
16. A.A. Osipov, B. Hiller, Phys. Lett. B **539**, 76 (2002) [hep-ph/0204182]
17. A.A. Osipov, B. Hiller, Eur. Phys. J. C **35**, 223 (2004) [hep-th/0307035]
18. A.A. Osipov, B. Hiller, V. Bernard, A.H. Blin, to be published in Ann. Phys. (New York) (2006) [hep-ph/0507226]
19. A.A. Osipov, B. Hiller, J. da Providência, Phys. Lett. B **634**, 48 (2006) [hep-ph/0508058]
20. S. Gasiorowicz, D.A. Geffen, Rev. Mod. Phys. **61**, 531 (1969)
21. S. Wolfram, The mathematica book, 5th ed. (Wolfram Media, 2003)
22. A.A. Osipov, B. Hiller, Phys. Lett. B **515**, 458 (2001) [hep-th/0104165]; A.A. Osipov, B. Hiller, Phys. Rev. D **64**, 087701 (2001) [hep-th/0106226]
23. A.A. Osipov, B. Hiller, Phys. Rev. D **63**, 094009 (2001) [hep-ph/0012294]
24. W. Pauli, F. Vilar, Rev. Mod. Phys. **21**, 434 (1949)
25. A.A. Osipov, B. Hiller, Phys. Rev. D **62**, 114013 (2000) [hep-ph/0007102]
26. M. Gell-Mann, R.J. Oakes, B. Renner, Phys. Rev. **175**, 2195 (1968)
27. Particle Data Group, S. Eidelman et al., Phys. Lett. B **592**, 1 (2004)
28. M. Jamin, Phys. Lett. B **538**, 71 (2002)
29. N.A. Törnqvist, Eur. Phys. J. C **11**, 359 (1999) [hep-ph/9905282]
30. M. Napsuciale, S. Rodriguez, Int. J. Mod. Phys. A **16**, 3011 (2001) [hep-ph/0204149]; M. Napsuciale, A. Wirzba, M. Kirchbach, nucl-th/0105055; M. Napsuciale, hep-ph/0204170
31. J. Schechter, Y. Ueda, Phys. Rev. D **3**, 2874 (1971); R. Delbourgo, M.D. Scadron, Int. J. Mod. Phys. A **13**, 657 (1998) [hep-ph/9807504]
32. G. 't Hooft, hep-th/9903189
33. E. van Beveren, T.A. Rijken, K. Metzger, C. Dullemond, G. Rupp, J.E. Ribeiro, Z. Phys. C **30**, 615 (1986); E. van Beveren, G. Rupp, N. Petropoulos, F. Kleefeld, Effective theories of low energy QCD. In 2nd Int. Workshop Hadron Physics, Coimbra, Portugal [AIP Conf. Proc. **660**, 353 (2003)]
34. J.A. Oller, E. Oset, J.R. Peláez, Phys. Rev. D **59**, 074001 (1999); Erratum, Phys. Rev. D **60**, 099906 (1999) [hep-ph/9804209]; Phys. Rev. Lett. **80**, 3452 (1988) [hep-ph/9803242]
35. D. Black, M. Harada, J. Schechter, Phys. Rev. Lett. **88**, 181603 (2002) [hep-ph/0202069]