A CATEGORICAL INVARIANT OF FLOW EQUIVALENCE OF SHIFTS

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Abstract. We prove that the Karoubi envelope of a shift — defined as the Karoubi envelope of the syntactic semigroup of the language of blocks of the shift — is, up to natural equivalence of categories, an invariant of flow equivalence. More precisely, we show that the action of the Karoubi envelope on the Krieger cover of the shift is a flow invariant. An analogous result concerning the Fischer cover of a synchronizing shift is also obtained. From these main results, several flow equivalence invariants — some new and some old — are obtained. We also show that the Karoubi envelope is, in a natural sense, the best possible syntactic invariant of flow equivalence of sofic shifts.

Another application concerns the classification of Markov-Dyck and Markov-Motzkin shifts: it is shown that, under mild conditions, two graphs define flow equivalent shifts if and only if they are isomorphic.

Shifts with property $(\mathcal{A})$ and their associated semigroups, introduced by Wolfgang Krieger, are interpreted in terms of the Karoubi envelope, yielding a proof of the flow invariance of the associated semigroups in the cases usually considered (a result recently announced by Krieger), and also a proof that property $(\mathcal{A})$ is decidable for sofic shifts.

Contents

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1. Introduction

Two discrete-time dynamical systems are *flow equivalent* if their suspension flows (or mapping tori) are equivalent. For symbolic dynamical systems, Parry and Sullivan characterized flow equivalence as the equivalence relation between shifts generated by conjugacy and a non-symmetric relation which at present is called *symbol expansion* [52]; see also [46, Section 13.7] and [2,4]. Special attention has been given to the classification of shifts of finite type up to flow equivalence; in that context, complete and decidable algebraic invariants were obtained in the irreducible case [23]. Complete algebraic invariants were also obtained in the reducible case [9, 30]. The problem of classifying sofic shifts up to flow equivalence is far more difficult [2, 10, 31].

The role of the syntactic semigroup of the language of finite blocks of a shift $\mathscr{X}$, which we call the *syntactic semigroup of* $\mathscr{X}$, has been considered in the literature [5–7,12,13,33,34] essentially in the context of (strictly) sofic shifts. In [16], one finds a characterization of the abstract semigroups which are the syntactic semigroup of an irreducible sofic shift.

For a semigroup S, let $\mathbb{K}(S)$ be the Karoubi envelope (also known as Cauchy completion or idempotent splitting) of S. It is a certain small category that plays a crucial role in finite semigroup theory thanks in part to the Delay Theorem of Tilson (see [61], where $\mathbb{K}(S)$ is denoted by S_E.)

For a shift $\mathscr{X}$, let $S(\mathscr{X})$ be its syntactic semigroup. In this paper we prove that the Karoubi envelope $\mathbb{K}(S(\mathscr{X}))$ (more briefly denoted $\mathbb{K}(\mathscr{X})$) is, up to equivalence of categories, a flow equivalence invariant of $\mathscr{X}$. This says, in a sense to be made more precise, that $\mathscr{X}$ determines $S(\mathscr{X})$ up to Morita equivalence. We also show that among sofic shifts this is the best possible syntactic invariant of flow equivalence, as every flow equivalence invariant of sofic shifts which is also invariant under isomorphism of syntactic semigroups is shown to factor through the equivalence relation identifying shifts with equivalent Karoubi envelopes.

The category $\mathbb{K}(\mathscr{X})$ is of little use when dealing with shifts of finite type. Indeed, a shift of finite type is conjugate with an edge shift, and it is easy to see that if $\mathscr{X}$ is an irreducible edge shift then $S(\mathscr{X})$ is isomorphic to a Brandt semigroup B_n for some n [14, Remark 2.23]. Unfortunately, all finite Brandt semigroups have equivalent Karoubi envelopes. However, for other classes, including strictly sofic shifts, we do obtain interesting results.

We also investigate the actions of the category $\mathbb{K}(\mathscr{X})$ on the Krieger cover of $\mathscr{X}$ and, if $\mathscr{X}$ is synchronizing, on its Fischer cover. We show that these actions are invariant under flow equivalence. As seen in Section 5, this enables a new proof of the invariance under flow equivalence of the proper communication graph of a sofic shift (a result from [2]).

In Section 6, we provide examples of pairs of almost finite type shifts $\mathscr{X}$ and $\mathscr{Y}$ such that $\mathbb{K}(\mathscr{X})$ and $\mathbb{K}(\mathscr{Y})$ are not equivalent, whereas other flow equivalence invariants fail to separate them. In the first of these examples we use a labeled poset, derived from Green's relations on $S(\mathscr{X})$, which was shown in [12] to be a conjugacy invariant of sofic shifts. This is a more refined version of an invariant previously considered in [6].

In Section 7 we see further examples of how the Karoubi envelope can be used to show that some classes of shifts are closed under flow equivalence. In particular we deduce again that the class of almost finite type shifts is stable under flow equivalence, a result from [24].

The Karoubi envelope is applied in Section 8 to classify up to flow equivalence two classes of non-sofic shifts, the Markov-Dyck and the Markov-Motzkin shifts of Krieger and Matsumoto [43], thus reproving and generalizing the classification of Dyck shifts, first obtained in [50].

The Karoubi envelope is also used in Section 9 to give a new perspective on the class of shifts with property $(\mathcal{A})$, introduced by Wolfgang Krieger in [40], and also studied in [28, 29, 42]. The semigroup which Krieger associated to each shift with property $(\mathcal{A})$ is shown to be encapsulated in the Karoubi envelope. From that result one deduces the invariance, recently announced by Krieger, of such semigroups under flow equivalence, assuming the density of a special subset of the shift, an assumption which is usually made when studying property $(\mathcal{A})$ shifts. The interpretation in terms of the Karoubi envelope is also used to show that property $(\mathcal{A})$ is decidable for sofic shifts and to construct the first examples of sofic shifts without propert $(\mathcal{A})$.

The poset of subsynchronizing subshifts of a sofic shift was shown in [34] to be a conjugacy invariant of sofic shifts, with the aim of studying the structure of reducible sofic shifts. In Section 10 this invariance is recovered and extended to flow equivalence, using the Karoubi envelope.

The basic outline of the paper is as follows. We begin with two sections of preliminaries. The first contains some aspects of category theory and semigroup theory that we shall require in order to extract flow equivalence invariants from the category $\mathbb{K}(\mathscr{X})$. The second preliminary section is about symbolic dynamics. Section 4 states our main results. In Sections 5 to 10, consequences of our main results are explored, as well as relations with previous work, from the viewpoint of the classification of shifts up to flow equivalence. Although the principal motivation in this paper is the study of flow equivalence, we also deduce in Section 11 that the Karoubi envelope is an invariant of eventual conjugacy in the case of sofic shifts. The proofs of the main results are left to Sections 12 and 13.

There is also a short appendix completing an argument in the proof of Theorem 10.7, using technical tools from Section 12.

2. The Karoubi envelope of a semigroup

2.1. **Categorical preliminaries.** The reader is referred to [47] for basic notions from category theory. A *category* C consists of a class C_0 of *objects*, a class C_1 of *arrows* or *morphisms* and mappings $d, r\colon C_1 \longrightarrow C_0$ selecting the domain $d(f)$ and range $r(f)$ of each arrow f. In addition, there is an associative product on pairs of composable arrows $(f, g) \mapsto fg$, where composable means that $d(f) = r(g)$, and for each object $c \in C_0$, there is an identity arrow 1_c so that $1_c f = f$ and $g1_c = g$ when these compositions make sense. A category is said to be *small* if its objects and arrows form a set. The set of all arrows $f\colon c \longrightarrow d$ is denoted $C(c, d)$ and is called a *hom-set*.

A *functor* $F\colon C \longrightarrow D$ consists of a pair of mappings $F\colon C_0 \longrightarrow D_0$ and $F\colon C_1 \longrightarrow D_1$ preserving all the above structure, e.g., $F(1_c) = 1_{F(c)}$ and $F(fg) = F(f)F(g)$, etc. A *natural transformation* $\eta\colon F \Rightarrow G$ of functors $F, G\colon C \longrightarrow D$ is a family $\{\eta_c\}_{c\in C_0}$ of arrows of D such that $\eta_c\colon F(c) \longrightarrow G(c)$ and, for each arrow $f\colon c \longrightarrow c'$ of C, the diagram

$$\begin{tikzcd} F(c) \arrow[r, "F(f)"] \arrow[d, "\eta_c"'] & F(c') \arrow[d, "\eta_{c'}"] \\ G(c) \arrow[r, "G(f)"'] & G(c') \end{tikzcd}$$

commutes. The class of natural transformations $F \Rightarrow G$ forms a category in the obvious way and two functors are *isomorphic*, written $F \cong G$ if they are isomorphic in this category. Contravariant functors are defined similarly except that $F(fg) = F(g)F(f)$.

Two categories C and D are *equivalent* if there are functors $F\colon C \longrightarrow D$ and $G\colon D \longrightarrow C$ such that $FG \cong 1_D$ and $GF \cong 1_C$. Such a functor G is said to be a *quasi-inverse* of F. A functor $F\colon C \longrightarrow D$ between small categories is an equivalence (i.e., it has a quasi-inverse) if and only if it is fully faithful and essentially surjective. *Fully faithful* means bijective on hom-sets, whereas *essentially surjective* means that every object of D is isomorphic to an object of $F(C)$. The former is in accordance with the usual terminology for functors which are injective on hom-sets (the *faithful* functors) and for those surjective on hom-sets (the *full* functors.)

2.2. **The Karoubi envelope and Morita equivalence of semigroups.** An important notion in this paper is the *Karoubi envelope* $\mathbb{K}(S)$ (also known as *Cauchy completion* or *idempotent splitting*) of a semigroup S. It is a small category whose object set is the set $E(S)$ of idempotents of S. Morphisms in $\mathbb{K}(S)$ from f to e are represented by arrows $e \longleftarrow f$, with source on the right. All other categories will be treated as usual with arrows drawn from left to right. This is to keep our notation for arrows of $\mathbb{K}(S)$ consistent with [61]. A morphism $e \longleftarrow f$ is a triple (e, s, f) where $s \in eSf$. Note that $s \in eSf$ if and only if $s = esf$, because e and f are idempotents. Composition of morphisms is given by $(e, s, f)(f, t, g) = (e, st, g)$. The identity at e is (e, e, e). Note that the Karoubi envelope is functorial.

It is easy to show that idempotents e, f are isomorphic in $\mathbb{K}(S)$ if and only if there exist $x, x' \in S$ such that $xx'x = x$, $x'xx' = x'$, $x'x = e$ and $xx' = f$. In semigroup terms (which will be explained in more detail in Subsection 2.5), this says that e, f are $\mathcal{D}$-equivalent [58], whereas in analytic terms this corresponds to von Neumann-Murray equivalence.

If e is an idempotent of the semigroup S, then eSe is a monoid with identity e, which is called the *local monoid of S at e*. The *local monoid of a category C at an object c* is the endomorphism monoid of c in C. The local monoids of S correspond to the local monoids of $\mathbb{K}(S)$, more precisely, eSe and $\mathbb{K}(S)(e, e)$ are isomorphic for every $e \in E(S)$.

An element s of a semigroup S has *local units e and f*, where e and f are idempotents of S, if $s = esf$. The set $LU(S) = E(S)SE(S)$ of elements of S with local units is a subsemigroup of S. If $LU(S) = S$, then we say that S has *local units*. In general, $LU(S)$ is the largest subsemigroup of S which has local units (it may be empty.) Clearly $\mathbb{K}(S) = \mathbb{K}(LU(S))$ and so the Karoubi envelope does not distinguish between these two semigroups. Talwar defined in [60] a notion of Morita equivalence of semigroups with local units in terms of equivalence of certain categories of actions. It was shown in [25, 45] that semigroups S and T with local units are Morita equivalent if and only if $\mathbb{K}(S)$ and $\mathbb{K}(T)$ are equivalent categories. Thus we shall say that semigroups S and T are *Morita equivalent up to local units* if $\mathbb{K}(S)$ and $\mathbb{K}(T)$ are equivalent categories, or in other words if $LU(S)$ is Morita equivalent to $LU(T)$. In this paper, we will show that flow equivalent shifts have syntactic semigroups that are Morita equivalent up to local units.

If C is a category, then we shall say an assignment $S \mapsto F(S)$ of an object $F(S)$ of C to each semigroup S is a *Karoubi invariant* if $F(S) \cong F(T)$ whenever $\mathbb{K}(S)$ and $\mathbb{K}(T)$ are equivalent.

2.3. **Categories with zero.** A *semigroup with zero* is a semigroup S with an element 0 such that $0s = 0 = s0$ for all $s \in S$. The element 0 is unique.

A *pointed set* (X, x) is a set X together with a distinguished element $x \in X$ called the *base point*. A morphism of pointed sets $f\colon (X, x) \longrightarrow (Y, y)$ is a function $f\colon X \longrightarrow Y$ with $f(x) = y$. We will customarily denote the base point by 0 if confusion cannot arise. The category of pointed sets will be denoted $\mathbf{Set}_0$.

A *category with zero* is a category C enriched over the monoidal category of pointed sets. What this means concretely is that, for all objects c, d of C, there is a zero morphism $0_{c,d} \in C(c, d)$ such that for all $f\colon c' \longrightarrow c$ and $g\colon d \longrightarrow d'$ one has $0_{c,d}f = 0_{c',d}$ and $g0_{c,d} = 0_{c,d'}$. It is easy to check that the zero morphisms are uniquely determined and so, from now on, we will drop the subscripts on zero morphisms when convenient.

The category of pointed sets is an example of a category with 0, where the zero map sends each element to the base point. The most important example for us is the case where S is a semigroup with zero and C is the Karoubi envelope $\mathbb{K}(S)$. Then $0_{e,f} = (f, 0, e)$ is the zero morphism of $\mathbb{K}(S)(e, f)$. An object c of a category with zero is said to be *trivial* if $1_c = 0_c$. Notice then that the only morphisms into and out of a trivial object are zero morphisms.

Note that if F is a full functor between categories with zero, then $F(0) = 0$. This enables us to register the following remark, for later reference.

Remark 2.1. If F is a fully faithful functor between categories with zero then $F(x) = 0$ if and only if $x = 0$.

In general, a morphism of categories with zero should be defined as a functor preserving zero morphisms.

2.4. **Actions of $\mathbb{K}(S)$.** Recall that a (right) *action* of a semigroup S on a set Q is a function $\mu\colon Q \times S \longrightarrow S$, with notation $\mu(q, s) = q \cdot s = qs$, such that $q \cdot (st) = (q \cdot s) \cdot t$. The action defines a function φ from S to the monoid Q^Q of transformations of Q, given by $\varphi(s)(q) = q \cdot s$. An action of a semigroup S with zero on a *pointed set* $(Q, 0)$ is an action such that $0s = 0$ and $q0 = 0$ for all $s \in S$ and $q \in Q$. Actions on pointed sets are essentially the same thing as actions by partial functions. Morphisms between actions on pointed sets are defined in the obvious way.

A (right) *action* of a small category C with zero on a pointed set Q is a contravariant functor $\mathbb{A}\colon C \longrightarrow \mathbf{Set}_0$, preserving 0, such that $\mathbb{A}(c)$ is a pointed subset of Q, for every object c of C. If $s\colon c \longrightarrow d$ is a morphism of C, we use the notation $q \cdot s$ for $\mathbb{A}(s)(q)$, where $q \in \mathbb{A}(d)$, and the notation $\mathbb{A}(d) \cdot s$ for the image of the function $\mathbb{A}(s)\colon \mathbb{A}(d) \longrightarrow \mathbb{A}(c)$. The notations $q \cdot s$ and $\mathbb{A}(d) \cdot s$ may be simplified to qs and $\mathbb{A}(d)s$.

In concrete terms this means that $\mathbb{A}(c)$ is a subset of Q containing 0 for each object of $c \in C$; if $s\colon c \longrightarrow d$ is an arrow then we have a map $\mathbb{A}(d) \longrightarrow \mathbb{A}(c)$ given by $q \mapsto qs$ such that $0s = 0$ and $q0 = 0$ for all s, q and where the obvious associativity and identity actions hold.

We can define a category $\mathbf{Act}_0$ whose objects consist of pairs $(\mathbb{A}, C)$ where C is a category with zero and $\mathbb{A}$ is an action of C on a pointed set. A morphism $F\colon (\mathbb{A}, C) \longrightarrow (\mathbb{B}, D)$ is a pair (η, F) where $F\colon C \longrightarrow D$ is a zero-preserving functor and $\eta\colon \mathbb{A} \longrightarrow \mathbb{B} \circ F$ is a natural transformation. Two actions $(\mathbb{A}, C)$ and $(\mathbb{A}', D)$ of categories with zero on pointed sets are *equivalent*, written $\mathbb{A} \sim \mathbb{A}'$, if they are isomorphic in the category $\mathbf{Act}_0$.

Definition 2.2. Consider an action of a semigroup S with zero on a pointed set Q. Let $\mathbb{A}_Q$ be the action of $\mathbb{K}(S)$ on Q such that $\mathbb{A}_Q(e) = Qe$ for every object e of $\mathbb{K}(S)$, and such that $q \cdot (e, s, f) = q \cdot s$ for $q \in Qe$.

It is easy to check the construction of Definition 2.2 is functorial in an appropriate category, yielding that isomorphic actions of semigroups produce isomorphic actions of the respective Karoubi envelopes

Let C be a category. An assignment of an object $F(Q, S)$ of C to each action of a semigroup S with zero on a pointed set Q is said to be a *Karoubi invariant* of the action if $F(Q, S)$ is isomorphic to $F(Q', T)$ whenever the actions $(\mathbb{K}(S), \mathbb{A}_Q)$ and $(\mathbb{K}(T), \mathbb{A}_{Q'})$ are equivalent.

2.5. **Green's relations.** Throughout this paper we use basic notions from semigroup theory that can be found in standard texts [11, 44, 58]. Green's (equivalence) relations, which we next recall, are among them. The relation $\mathcal{J}$ is defined on a semigroup S by putting $s \mathrel{\mathcal{J}} t$ if s and t generate the same two-sided principal ideal, that is, if $S^1sS^1 = S^1tS^1$, where S^1 denotes the monoid obtained from S adjoining an identity. Similarly, one defines the $\mathcal{R}$- and $\mathcal{L}$-relations, by replacing two-sided ideals with right (respectively, left) ideals. The intersection of the equivalence relations $\mathcal{R}$ and $\mathcal{L}$ is denoted by $\mathcal{H}$. The maximal subgroups of S are the $\mathcal{H}$-classes containing idempotents. Finally, Green's relation $\mathcal{D}$ is defined by $s \mathrel{\mathcal{D}} t$ if and only if there exists $u \in S$ with $s \mathrel{\mathcal{R}} u \mathrel{\mathcal{L}} t$. This is known to be equivalent to the existence of $v \in S$ with $s \mathrel{\mathcal{L}} v \mathrel{\mathcal{R}} t$. This is the smallest equivalence relation containing $\mathcal{L}$ and $\mathcal{R}$. Details can be found in [11]. For finite semigroups, one has $\mathcal{D} = \mathcal{J}$ [58, Appendix A].

An element s of a semigroup S is *regular* if $s = sxs$ for some $x \in S$. A $\mathcal{D}$-class contains regular elements if and only if it contains an idempotent, if and only if all its elements are regular. A $\mathcal{D}$-class with regular elements is called *regular*.

Let H be an $\mathcal{H}$-class of S. The set $T = \{x \in S^1 : xH \subseteq H\}$ is a submonoid of S^1, called the *left stabilizer* of H. The quotient of T by its left action on H is a group known as the *Schützenberger group of* H. Exchanging right and left, one obtains an isomorphic group. If H is a group (which occurs if and only if it contains an idempotent), then it is isomorphic to its Schützenberger group. Two $\mathcal{H}$-classes contained in the same $\mathcal{D}$-class have isomorphic Schützenberger groups, hence the expression *Schützenberger group of a $\mathcal{D}$-class* is meaningful. See [11] for details.

The following lemma concerning the Karoubi envelope $\mathbb{K}(S)$ of a semigroup is well known and easy to prove [45, 58].

Lemma 2.3. *Two objects e and f of $\mathbb{K}(S)$ are isomorphic if and only if e and f are $\mathcal{D}$-equivalent in S. Moreover, the automorphism group of e is isomorphic to the Schützenberger group of the $\mathcal{D}$-class of e.*

The following lemma will be useful several times.

Lemma 2.4. *Suppose that u, v are two elements of a semigroup S such that $u = zt$ and $v = tz$ with $z, t \in S$. If u is an idempotent, then so is v^2. Moreover, $u \mathcal{D} v$.*

Proof. Trivially, $v^4 = tu^3v = tuz = v^2$ and so v^2 is idempotent. Next, let $s = uz$ and observe that $st = uzt = u^2 = u$. Also, $ts = tztz = v^2$ and $zv^2 = ztztz = u^2z = uz = s$. Thus we have $u \mathcal{R} s \mathcal{L} v^2$. □

3. Symbolic dynamics

3.1. **Shifts.** A good reference for the notions that we shall use here from symbolic dynamics is [46]. To make the paper reasonably self-contained and to introduce notation, we recall some basic definitions.

Let A be a finite alphabet, and consider the set $A^{\mathbb{Z}}$ of all bi-infinite sequences over A. The *shift* on $A^{\mathbb{Z}}$ is the homeomorphism $\sigma_A \colon A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z}}$ (or just σ) defined by $\sigma_A((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$. We endow $A^{\mathbb{Z}}$ with the product topology with respect to the discrete topology on A. In particular, $A^{\mathbb{Z}}$ is a compact totally disconnected space (we include the Hausdorff property in the definition of compact.)

We assume henceforth that all alphabets are finite. A *symbolic dynamical system* is a non-empty closed subset $\mathscr{X}$ of $A^{\mathbb{Z}}$, for some alphabet A, such that $\sigma(\mathscr{X}) = \mathscr{X}$. Symbolic dynamical systems are also known as *shift spaces*, *subshifts*, or, more plainly, just *shifts*. We shall generally prefer the latter option except when emphasizing that one shift is a subshift of another.

We can consider the category of shifts, whose objects are the shifts and where a morphism between two shifts $\mathscr{X} \subseteq A^{\mathbb{Z}}$ and $\mathscr{Y} \subseteq B^{\mathbb{Z}}$ is a continuous function $\Phi \colon \mathscr{X} \longrightarrow \mathscr{Y}$ such that $\Phi \circ \sigma_A = \sigma_B \circ \Phi$. In this category, an isomorphism is called a *conjugacy*. Isomorphic shifts are said to be *conjugate*.

A *block* of $(x_i)_{i\in\mathbb{Z}} \in A^{\mathbb{Z}}$ is a word of the form $x_i x_{i+1} \cdots x_{i+n}$ (briefly denoted by $x_{[i,i+n]}$), where $i \in \mathbb{Z}$ and $n \geq 0$. If $\mathscr{X}$ is a subset of $A^{\mathbb{Z}}$ then $L(\mathscr{X})$ denotes the set of blocks of elements of $\mathscr{X}$. If $\mathscr{X}$ is a subshift of $A^{\mathbb{Z}}$ and $x \in A^{\mathbb{Z}}$, then $x \in \mathscr{X}$ if and only if $L(\{x\}) \subseteq L(\mathscr{X})$ [46, Corollary 1.3.5].

If x is a periodic point with period $n \geq 1$ (that is, if $\sigma^n(x) = x$), then we may represent x by u^∞, where $u = x_{[0,n-1]}$.

3.2. **The syntactic semigroup of a shift.** We shall make use of several well-known fundamental ideas and facts about the interplay between finite automata, formal languages and semigroups which, with some variations, can be found in several books, such as [20, 57].

The free semigroup and the free monoid over an alphabet A are denoted by A^+ and A^*, respectively. Recall that a subset of A^+ is in this context called a *language*. If A is an alphabet, let $A_0^+ = A^+ \cup \{0\}$ be the free semigroup on A with zero. The multiplication of A^+ is extended to make 0 a zero element. Let $L \subseteq A^+$ be a language and let $u \in A_0^+$. The *context of u in L* is the set $[u]_L = \{(x, y) \in A^* \colon xuy \in L\}$. Of course, $[0]_L = \emptyset$. The relation $\equiv_L$ on A_0^+ such that $u \equiv_L v$ if and only if $[u]_L = [v]_L$ is a semigroup congruence, and the quotient semigroup $A_0^+/{\equiv_L}$ is the *syntactic semigroup (with zero) of* L, denoted $S(L)$. The quotient homomorphism $A_0^+ \longrightarrow S(L)$ is the *syntactic homomorphism*, denoted δ_L. Note that the class of 0 is the zero of $S(L)$. Since $u \equiv_L v$ if and only if $[u]_L$ and $[v]_L$ are equal, we may identify the $\equiv_L$-equivalence class of u with $[u]_L$.

An onto homomorphism $\varphi\colon A_0^+ \longrightarrow S$ *recognizes* the language L of A^+ if $L = \varphi^{-1}(P)$ for some subset P of S, in which case we also say that S recognizes L, a property equivalent to S being isomorphic to a quotient of A_0^+ by a congruence that saturates L (a semigroup is *saturated* by a congruence when it is a union of some of its congruence classes.) It turns out that the syntactic congruence $\equiv_L$ is the greatest congruence (for the inclusion) that saturates L. This means that if a semigroup S recognizes L as a subset of A_0^+, then $S(L)$ is a homomorphic image of S.

For a subshift $\mathscr{X}$ of $A^{\mathbb{Z}}$, we use the notations $\delta_{\mathscr{X}}$, $S(\mathscr{X})$, $LU(\mathscr{X})$ instead of $\delta_{L(\mathscr{X})}$, $S(L(\mathscr{X}))$ and $LU(S(\mathscr{X}))$ respectively. Also, instead of $[u]_{L(\mathscr{X})}$, we may use $[u]_{\mathscr{X}}$ or simply $[u]$ if no confusion arises. We say that $S(\mathscr{X})$ is the *syntactic semigroup of* $\mathscr{X}$. If $\mathscr{X} \subseteq A^{\mathbb{Z}}$ and $A \subseteq B$, then the syntactic semigroups of $\mathscr{X}$ viewed as a subset of $A^{\mathbb{Z}}$ and of $B^{\mathbb{Z}}$ coincide.

Remark 3.1. One has $u \in A_0^+ \setminus L(\mathscr{X})$ if and only if $[u] = \emptyset$. In particular, if $[u]$ is a non-zero idempotent, then the periodic point u^∞ belongs to $\mathscr{X}$.

Let S be a semigroup with zero such that $SsS \neq \{0\}$ for every $s \in S\setminus\{0\}$. We say such a semigroup is *prolongable*. Suppose S is generated by a finite set A, and let $\varphi\colon A^+ \longrightarrow S$ be an onto homomorphism S. Then $L = \varphi^{-1}(S \setminus \{0\})$ is clearly a factorial prolongable language, and so there is a unique subshift $\mathscr{X}_\varphi$ of $A^{\mathbb{Z}}$ such that $L = L(\mathscr{X}_\varphi)$ (cf. [46, Proposition 1.3.4].) We say that $\mathscr{X}_\varphi$ *is the shift induced by* φ, or, more vaguely, *induced by* S.

Let us call a congruence *trivial* if it is the equality relation. A semigroup S with zero is 0-*disjunctive* if the greatest congruence saturating $\{0\}$, (equivalently, saturating $S\setminus\{0\}$) is the trivial one. The next simple lemma concerns a well known property about syntactic semigroups (cf. [44, Proposition 5.3].)

Lemma 3.2. *The syntactic semigroup $S(\mathscr{X})$ of a shift $\mathscr{X}$ is 0-disjunctive. Conversely, if S is a prolongable semigroup with zero which is 0-disjunctive, then S is the syntactic semigroup of every shift induced by S.*

3.3. **Labeled graphs and sofic shifts.**

3.3.1. *Labeled graphs.* In this paper, *graph* will always mean a multi-edge directed graph. By a *labeled graph (over an alphabet A)* we mean a pair (G,λ) consisting of a graph G and a map $\lambda\colon E(G) \longrightarrow A$, where $E(G)$ is the set of edges of G, such that if e and f are distinct edges with the same origin and the same terminus, then $\lambda(e) \neq \lambda(f)$. The letter $\lambda(e)$ is the *label* of e. The labeled graph (G,λ) is *right-resolving* if distinct edges with the same origin have distinct labels. It is *complete* if, for every vertex q and every letter $a \in A$, there is an edge starting in q with label a. Note that a complete, right-resolving labeled graph over A may be viewed as an action of A^+ over the set of vertices: the existence of an edge from q to r labeled a corresponds to the equality $q \cdot a = r$.

3.3.2. *Minimal automaton.* An *automaton* is a labeled graph (whose vertices in this context are often called *states*) together with a set I of *initial states* and a set F of *final states*. The automaton is *deterministic* if it is right-resolving and has a single initial state. The set of words labeling paths from I to F is the language *recognized* by the automaton. We view a labeled graph as an automaton in which all states are initial and final.

Consider a language $L \subseteq A^+$. Let u be an element of $A_0^* = A^* \cup \{0\}$, the free monoid with zero. The *right context of u in L* is the set $R_L(u) = \{w \in A^* \mid uw \in L\}$. Viewing L as a language over the alphabet $A \cup \{0\}$, we can consider its *minimal complete deterministic automaton*, which is the terminal object in the category of complete deterministic automatons, over $A \cup \{0\}$, recognizing L. This automaton, which we denote by $\mathfrak{M}(L)$, can be realized as follows: the states are the right contexts of L, the initial state is $R_L(1)$, the final states are the right contexts $R_L(u)$ such that $u \in L$, and the action of an element v of A_0^+ is given by $R_L(u) \cdot v = R_L(uv)$. A *sink* in a labeled graph is a vertex z such that all edges starting in z are loops; the vertex $R_L(0) = \emptyset$ is the unique sink of $\mathfrak{M}(L)$. The language L is *recognizable* if it can be recognized by a finite automaton; L is recognizable if and only if $\mathfrak{M}(L)$ is finite, if and only if $S(L)$ is finite [20, 57].

Note that $u, v \in A_0^+$ have the same action on the states of $\mathfrak{M}(L)$ if and only if $[u]_L = [v]_L$. In particular, we may consider the action of $S(L)$ on the set of states defined by $R_L(u) \cdot [v]_L = R_L(uv)$.

For a subshift $\mathscr{X}$ of $A^{\mathbb{Z}}$, we use the notations $\mathfrak{M}(\mathscr{X})$ and $R_{\mathscr{X}}(u)$ instead of $\mathfrak{M}(L(\mathscr{X}))$ and $R_{L(\mathscr{X})}(u)$.

3.3.3. *Sofic shifts.* A graph G is *essential* if the in-degree and the out-degree of each vertex is at least one. If $L(\mathscr{X})$, for a shift $\mathscr{X}$, is recognized by the essential labeled graph (G, λ), then $\mathscr{X}$ is the *shift presented by* (G, λ).

The shifts that can be presented by a finite labeled graph are called *sofic* [46, Chapter 3]. The sofic shifts are the shifts $\mathscr{X}$ such that $L(\mathscr{X})$ is recognizable. That is, $\mathscr{X}$ is sofic if and only if $S(\mathscr{X})$ is finite. The most studied class of sofic shifts is that of *finite type shifts* [46, Chapter 2]: a subshift $\mathscr{X}$ of $A^{\mathbb{Z}}$ is of finite type when $L(\mathscr{X}) = A^+ \setminus A^*WA^*$, for some finite subset W of A^+. An *edge shift* is a shift presented by a finite essential labeled graph (G, λ) such that the mapping λ is one-to-one. One of the characterizations of the shifts of finite type is that they are the shifts conjugate to edge shifts [46, Theorem 2.3.2].

A subshift $\mathscr{X}$ of $A^{\mathbb{Z}}$ is *irreducible* if, for all $u, v \in L(\mathscr{X})$, there is $w \in A^*$ such that $uwv \in L(\mathscr{X})$. A sofic shift is irreducible if and only if it can be presented by a strongly connected labeled graph [46, Proposition 3.3.11].

3.4. **Synchronizing shifts.** A word u of $L(\mathscr{X})$ is *synchronizing*[1] if $vu, uw \in L(\mathscr{X})$ implies $vuw \in L(\mathscr{X})$. An irreducible shift $\mathscr{X}$ is *synchronizing* if $L(\mathscr{X})$ contains a synchronizing word. Every irreducible sofic shift is synchronizing (cf. [22, Lemma 2] and [8, Proposition 3.1].) Also, a shift is of finite type if and only if there is some n such that every word of $L(\mathscr{X})$ of length at least n is synchronizing, in which case the shift is said to be an *n-step shift* [46, Theorem 2.1.8].

The following lemma about synchronizing words can be useful.

Lemma 3.3. *Let $\mathscr{X} \subseteq A^{\mathbb{Z}}$ be a shift for which $L(\mathscr{X})$ has a synchronizing word (e.g., a synchronizing shift.)*

(1) The union of $A^+ \setminus L(\mathscr{X})$ with the set of synchronizing words of $L(\mathscr{X})$ is an ideal of A^+.
(2) If u is synchronizing, then $[u]$ is idempotent if and only if $u^2 \in L(\mathscr{X})$.
(3) If v is synchronizing, then $uv \in L(\mathscr{X})$ implies $R_{\mathscr{X}}(uv) = R_{\mathscr{X}}(v)$.

Proof. Suppose that u is synchronizing and let $r \in A^+$. Then if $vru, ruw \in L(\mathscr{X})$, one has that $(vr)uw \in L(\mathscr{X})$ because u is synchronizing. Thus ru is synchronizing. Similarly, ur is synchronizing. This proves the first item.

Suppose that u maps to an idempotent of $S(\mathscr{X})$. Then trivially $u^2 \in L(\mathscr{X})$. For the converse, suppose that $u^2 \in L(\mathscr{X})$. If $v, w \in A^*$ with $vuw \in L(\mathscr{X})$, then because $vu, uu \in L(\mathscr{X})$, we have $vuu \in L(\mathscr{X})$. But then $vuu, uw \in L(\mathscr{X})$ implies that $vu^2w \in L(\mathscr{X})$. Conversely, if $vu^2w \in L(\mathscr{X})$, then $vu, uw \in L(\mathscr{X})$ and so $vuw \in L(\mathscr{X})$ because u is synchronizing.

The final statement follows because $vw \in L(\mathscr{X})$ if and only if $uvw \in L(\mathscr{X})$ by the definition of a synchronizing word. $\square$

The main results of this paper concern the Karoubi envelope of $S(\mathscr{X})$, which has at least one object, namely 0. The next proposition gives a sufficient condition for the existence of other objects; it is not necessary, as witnessed by the classes of shifts analyzed in Section 8.

Proposition 3.4. *Let $\mathscr{X}$ be a synchronizing shift. If u is a synchronizing word of $L(\mathscr{X})$, then $R_{\mathscr{X}}(u) = R_{\mathscr{X}}(1) \cdot e$ for some idempotent $e \in S(\mathscr{X}) \setminus \{0\}$.*

Proof. As $\mathscr{X}$ is irreducible, there is v such that $uvu \in L(\mathscr{X})$. Then $vu, uvu \in L(\mathscr{X})$ implies that $vuvu \in L(\mathscr{X})$. By Lemma 3.3, vu is synchronizing and $e = [vu]$ is a non-zero idempotent. Lemma 3.3 also yields $R_{\mathscr{X}}(u) = R_{\mathscr{X}}(vu) = R_{\mathscr{X}}(1) \cdot e$. $\square$

3.5. **Krieger and Fischer covers.** Denote the set of negative integers by $\mathbb{Z}^-$, and the set of non-negative integers by $\mathbb{N}$. Given an element $x = (x_i)_{i \in \mathbb{Z}^-}$ of $A^{\mathbb{Z}^-}$ and an element $y = (y_i)_{i \in \mathbb{N}}$ of $A^{\mathbb{N}}$, denote by $x.y$ the element $z = (z_i)_{i \in \mathbb{Z}}$ of $A^{\mathbb{Z}}$ for which $z_i = x_i$ if $i < 0$, and $z_i = y_i$ if $i \geq 0$.

[1] In [46], a synchronizing word is called an *intrinsically synchronizing* word. There is some diversity of terminology in the literature (cf. [2, Remark 2.6].)

If $x \in A^{\mathbb{Z}^-}$ and $u = a_1 \cdots a_n \in A^+$, with $a_i \in A$ when $1 \leq i \leq n$, then xu denotes the element of $A^{\mathbb{Z}^-}$ given by the left-infinite sequence $\cdots x_{-3}x_{-2}x_{-1}a_1 \cdots a_n$. Similarly, if $x \in A^{\mathbb{N}}$ then $ux \in A^{\mathbb{N}}$ is given by the right-infinite sequence $a_1 \cdots a_n x_0 x_1 x_2 \cdots$.

Let $C_{\mathscr{X}}(x) = \{y \in A^{\mathbb{N}} : x.y \in \mathscr{X}\}$, where $\mathscr{X}$ is a subshift of $A^{\mathbb{Z}}$ and $x \in A^{\mathbb{Z}^-}$. If $u \in A^+$, then $C_{\mathscr{X}}(x) = C_{\mathscr{X}}(z)$ implies $C_{\mathscr{X}}(xu) = C_{\mathscr{X}}(zu)$. This enables the following definition.

Definition 3.5 (Krieger cover)**.** Let $Q(\mathscr{X}) = \{C_{\mathscr{X}}(x) \mid x \in A^{\mathbb{Z}^-}\} \cup \{\emptyset\}$. Denote by $\mathfrak{K}^0(\mathscr{X})$ the right-resolving complete labeled graph over $A \cup \{0\}$, with vertex set $Q(\mathscr{X})$, defined by the action of A_0^+ on $Q(\mathscr{X})$ given by

$$C_{\mathscr{X}}(x) \cdot u = C_{\mathscr{X}}(xu) \quad \text{if } u \in A^+, \quad C_{\mathscr{X}}(x) \cdot 0 = \emptyset,$$

having $\emptyset$ as unique sink. The labeled graph $\mathfrak{K}(\mathscr{X})$ over A obtained from $\mathfrak{K}^0(\mathscr{X})$ by elimination of the vertex $\emptyset$ is the *right Krieger cover of* $\mathscr{X}$ (cf. [21, Definition 0.11].)

Krieger introduced in [39] this cover for sofic shifts only. If $\mathscr{X}$ is sofic, then the right Krieger cover of $\mathscr{X}$ embeds in the automaton $\mathfrak{M}(\mathscr{X})$ and it is computable [4, Section 4]. There are examples of synchronizing shifts whose Krieger graph is uncountable (cf., the example in the proof of [21, Corollary 1.3].) Hence, in the non-sofic case, the Krieger cover may not be a labeled subgraph of $\mathfrak{M}(\mathscr{X})$, which is always at most countable.

We next relate $S(\mathscr{X})$ with the Krieger cover.

Lemma 3.6. *Consider a subshift $\mathscr{X}$ of $A^{\mathbb{Z}}$, and let $u, v \in A^+$. Then $[u] \subseteq [v]$ if and only if $C_{\mathscr{X}}(xu) \subseteq C_{\mathscr{X}}(xv)$ for all $x \in A^{\mathbb{Z}^-}$.*

Proof. Assume that $[u] \subseteq [v]$ and suppose that $y \in C_{\mathscr{X}}(xu)$. Then we have $x_{[-n,-1]}uy_{[0,n]} \in L(\mathscr{X})$ for every $n \geq 1$. As $[u] \subseteq [v]$, we obtain $x_{[-n,-1]}vy_{[0,n]} \in L(\mathscr{X})$ for every $n \geq 1$, which means that $y \in C_{\mathscr{X}}(xv)$.

Next suppose that $[u] \nsubseteq [v]$ and suppose $wuz \in L(\mathscr{X})$ and $wvz \notin L(\mathcal{X})$. Then, as $L(\mathscr{X})$ is prolongable, we can find left infinite x and right infinite y with $xwu.zy \in \mathcal{X}$. Then $zy \in C_{\mathscr{X}}(xwu)$ and $zy \notin C_{\mathscr{X}}(xwv)$. $\square$

Hence, by Lemma 3.6, the action of $S(\mathscr{X})$ on the set of vertices of $\mathfrak{K}^0(\mathscr{X})$ given by $C_{\mathscr{X}}(x) \cdot [u] = C_{\mathscr{X}}(xu)$ is well defined and faithful.

In a graph G, a subgraph H is *terminal* if every edge of G starting in a vertex of H belongs to H. A *strongly connected component* of G is a maximal strongly connected subgraph of G. As a reference for the next definition, we give Definition 0.12 in [21], and the lines following it in [21], for justification.

Definition 3.7 (Fischer cover)**.** If $\mathscr{X}$ is a synchronizing shift, then $\mathfrak{K}(\mathscr{X})$ has a sole strongly connected terminal component. This component presents $\mathscr{X}$. It is called the *right Fischer cover of* $\mathscr{X}$, and we denote it by $\mathfrak{F}(\mathscr{X})$.

We denote by $\mathfrak{F}^0(\mathscr{X})$ the terminal complete labeled subgraph of $\mathfrak{K}^0(\mathscr{X})$ obtained from $\mathfrak{F}(\mathscr{X})$ by adjoining the sink state $\{\emptyset\}$, and by $Q_{\mathfrak{F}}(\mathscr{X})$ the vertex set of $\mathfrak{F}^0(\mathscr{X})$.

See [21, Theorem 2.16] and [46, Corollary 3.3.19] for characterizations of the Fischer cover.

The next result was shown in [7] for irreducible sofic shifts. The generalization for synchronizing shifts offers no additional difficulty.

Proposition 3.8. *Let $\mathscr{X}$ be a synchronizing shift. Then the labeled graph obtained from $\mathfrak{M}(\mathscr{X})$ by eliminating the sink vertex $\emptyset$ has a unique terminal strongly connected component, which is isomorphic with $\mathfrak{F}(\mathscr{X})$. Its vertices are the right-contexts of synchronizing words.*

We have defined the *right* Krieger and Fischer covers. The *left* Krieger and Fischer covers are defined analogously, changing directions when needed.

If $\mathcal{A} = (G, \lambda)$ is a right-resolving labeled graph with state set Q and alphabet A, then recall we write $qa = q'$ if there is an edge from q to q' labeled by a. The *transition semigroup* $S(\mathcal{A})$ is the semigroup of partial mappings of Q generated by the maps $a \mapsto qa$. We write Q^0 for $Q \cup \{0\}$ with 0 an adjoined sink state and define $qs = 0$ if qs was undefined and $0s = 0$ for all $s \in S(\mathcal{A})$. For every shift $\mathscr{X}$, and because the action of $S(\mathscr{X})$ on its Krieger cover $\mathfrak{K}(\mathscr{X})$ is faithful (Lemma 3.6), the transition semigroup of $\mathfrak{K}(\mathscr{X})$ is isomorphic to $S(\mathscr{X})$. One can also easily check that if $\mathscr{X}$ is synchronizing then the action of $S(\mathscr{X})$ on the Fischer cover of $\mathscr{X}$ is faithful: the proof given in [4, Proposition 4.8] for irreducible sofic shifts holds for every synchronizing shift. Therefore, if $\mathscr{X}$ is synchronizing then $S(\mathscr{X})$ is isomorphic to the transition semigroup of the Fischer cover $\mathfrak{F}(\mathscr{X})$.

3.6. **Conjugacy and Nasu's theorem.** A right-resolving labeled graph $\mathcal{A} = (G, \lambda)$ over an alphabet A is *bipartite* if there are partitions $Q = Q_1 \uplus Q_2$ of the states and $A = A_1 \uplus A_2$ of the alphabet such that all edges labeled by A_1 go from Q_1 to Q_2 and all edges labeled by A_2 go from Q_2 to Q_1.

Let $\mathcal{A}_1 = (G_1, \lambda_1)$ be the right-resolving labeled graph over A_1A_2 with state set Q_1 obtained by turning each path of length 2 from Q_1 to itself into an edge (labeled by the product of the labels of the two edges) and define $\mathcal{A}_2 = (G_2, \lambda_2)$ with alphabet A_2A_1 and state set Q_2, analogously. We call $\mathcal{A}_1, \mathcal{A}_2$ the *components* of $\mathcal{A}$.

Write $\mathcal{A} \sim \mathcal{B}$ if there is a bipartite right-resolving labeled graph with components $\mathcal{A}, \mathcal{B}$ and let $\simeq$ be the equivalence relation on right-resolving labeled graphs generated by $\sim$.

The following fundamental result is in [51], where it is stated for sofic shifts but is well known to apply to any shift.

Theorem 3.9 (Nasu 1986)**.** *Let $\mathscr{X}_1, \mathscr{X}_2$ be shifts with Krieger covers $\mathfrak{K}(\mathscr{X}_1)$ and $\mathfrak{K}(\mathscr{X}_2)$, respectively. Then $\mathscr{X}_1$ and $\mathscr{X}_2$ are conjugate if and only if $\mathfrak{K}(\mathscr{X}_1) \simeq \mathfrak{K}(\mathscr{X}_2)$.*

3.7. **Flow equivalence.** Fix an alphabet A and a letter α of A. Let $\diamond$ be a letter not in A. Denote by B the alphabet $A \cup \{\diamond\}$. The *symbol expansion of A^* associated to α* is the unique monoid homomorphism $\mathcal{E} \colon A^* \longrightarrow B^*$ such that $\mathcal{E}(\alpha) = \alpha\diamond$ and $\mathcal{E}(a) = a$ for all $a \in A \setminus \{\alpha\}$. Note that $\mathcal{E}$ is injective.

The *symbol expansion of a subshift* $\mathscr{X}$ *of* $A^{\mathbb{Z}}$ *relatively to* α is the least subshift $\mathscr{X}'$ of $B^{\mathbb{Z}}$ such that $L(\mathscr{X}')$ contains $\mathcal{E}(L(\mathscr{X}))$. A *symbol expansion of* $\mathscr{X}$ is a symbol expansion of $\mathscr{X}$ relatively to some letter.

The mapping $\mathcal{E}$ admits the following natural extension of its domain and range. If $x \in A^{\mathbb{Z}^-}$ and $y \in A^{\mathbb{N}}$, then $\mathcal{E}(x)$ and $\mathcal{E}(y)$ are respectively the elements of $B^{\mathbb{Z}^-}$ and $B^{\mathbb{N}}$ given by $\mathcal{E}(x) = \dots \mathcal{E}(x_{-3})\,\mathcal{E}(x_{-2})\,\mathcal{E}(x_{-1})$ and $\mathcal{E}(y) = \mathcal{E}(y_0)\,\mathcal{E}(y_1)\,\mathcal{E}(y_2)\dots$. Moreover, $\mathcal{E}(x.y)$ denotes $\mathcal{E}(x).\mathcal{E}(y) \in B^{\mathbb{Z}}$. Note that $\mathscr{X}'$ is the least subshift of $B^{\mathbb{Z}}$ containing $\mathcal{E}(\mathscr{X})$.

Flow equivalence is the least equivalence relation between shifts containing the conjugacy and symbol expansion relations. The classes of finite type shifts, of sofic shifts, and of irreducible shifts are all easily seen to be closed under flow equivalence. See [46, Section 13.6] for motivation for studying flow equivalence. Here, we just remark that the original definition of flow equivalence (that two shifts are flow equivalent if their suspension flows are topologically equivalent) was proved in [52] to be equivalent to the one we use, explicitly for finite type shifts, but as pointed out in [49, Lemma 2.1], implicitly for all shifts. See also [53, page 87].

4. Statement of the main results

In this section we state the main results of this paper, deferring proofs to the final sections.

Definition 4.1 (Karoubi envelope of a shift)**.** Let $\mathscr{X}$ be a shift. We define the *Karoubi envelope* $\mathbb{K}(\mathscr{X})$ *of* $\mathscr{X}$ to be the Karoubi envelope $\mathbb{K}(S(\mathscr{X}))$ of its syntactic semigroup.

Note that the category $\mathbb{K}(\mathscr{X})$ is a category with zero, since $S(\mathscr{X})$ is a semigroup with zero. Our principal result is that the natural equivalence class of $\mathbb{K}(\mathscr{X})$ is a flow equivalence invariant of $\mathscr{X}$.

Theorem 4.2. *If* $\mathscr{X}$ *and* $\mathscr{Y}$ *are flow equivalent shifts, then the categories* $\mathbb{K}(\mathscr{X})$ *and* $\mathbb{K}(\mathscr{Y})$ *are equivalent, i.e.,* $S(\mathscr{X})$ *and* $S(\mathscr{Y})$ *are Morita equivalent up to local units. Hence every Karoubi invariant of* $S(\mathscr{X})$ *is a flow equivalence invariant of* $\mathscr{X}$*.*

We remark that if $\mathscr{X}$ and $\mathscr{Y}$ are sofic shifts given by presentations, then one can effectively determine whether $\mathbb{K}(\mathscr{X})$ is equivalent to $\mathbb{K}(\mathscr{Y})$. This is because these categories are finite and effectively constructible and so one can in principle check all functors between them and see if there is one which is an equivalence.

For a shift $\mathscr{X}$, denote by $\mathbb{A}_{\mathscr{X}}$ the action $\mathbb{A}_{Q(\mathscr{X})}$ arising from the action of $S(\mathscr{X})$ on $Q(\mathscr{X})$. The reader is referred back to Definition 2.2 for the notation.

Theorem 4.3. *If* $\mathscr{X}$ *and* $\mathscr{Y}$ *are flow equivalent shifts, then the actions* $(\mathbb{A}_{\mathscr{X}}, \mathbb{K}(\mathscr{X}))$ *and* $(\mathbb{A}_{\mathscr{Y}}, \mathbb{K}(\mathscr{Y}))$ *are equivalent. Hence any Karoubi invariant of the action of* $S(\mathscr{X})$ *on* $Q(\mathscr{X})$ *is an invariant of flow equivalence.*

The proofs of Theorems 4.2 and 4.3 are carried out in Section 12.

Remark 4.4. Recall from Subsection 3.5 the definition of right Fischer cover when $\mathscr{X}$ is a synchronizing shift. In particular, $Q_{\mathfrak{F}}(\mathscr{X})$ is the unique minimal $S(\mathscr{X})$-invariant subset of $Q(\mathscr{X})$ strictly containing the sink state.

Therefore, the action of $S(\mathscr{X})$ on $Q(\mathscr{X})$ restricts in a natural way to an action of $S(\mathscr{X})$ on $Q_{\mathfrak{F}}(\mathscr{X})$, denoted $\mathbb{A}^{\mathfrak{F}}_{\mathscr{X}}$.

Theorem 4.5. *If $\mathscr{X}$ and $\mathscr{Y}$ are flow equivalent synchronizing shifts, then the actions $(\mathbb{A}^{\mathfrak{F}}_{\mathscr{X}}, \mathbb{K}(\mathscr{X}))$ and $(\mathbb{A}^{\mathfrak{F}}_{\mathscr{Y}}, \mathbb{K}(\mathscr{Y}))$ are equivalent. Hence each Karoubi invariant of the action of $S(\mathscr{X})$ on $Q_{\mathfrak{F}}(\mathscr{X})$ is an invariant of flow equivalence.*

The proof of Theorem 4.5 is deferred to Section 12.3.

If $\mathscr{X}$ does not contain periodic points, for instance, if $\mathscr{X}$ is minimal non-periodic (cf. [46, Section 13.7]), then 0 is the unique idempotent of $S(\mathscr{X})$ (cf. Remark 3.1), and so in such cases our main results have no applications. On the other hand, they do have meaningful consequences for sofic, synchronizing, and other classes of shifts. In the next few sections we examine some of these consequences.

Before doing that, we show that the Karoubi envelope of a sofic shift is a universal syntactic invariant of flow equivalence of sofic shifts. But first, we need a suitable formalization of what this means.

An equivalence relation ϑ on the class of sofic shifts is an *invariant of flow equivalence* if whenever $\mathscr{X}$ and $\mathscr{Y}$ are flow equivalent sofic shifts, one has $\mathscr{X} \ \vartheta \ \mathscr{Y}$. For instance, our main theorem says that having naturally equivalent Karoubi envelopes is an invariant of flow equivalence.

Intuitively, a syntactic invariant of a sofic shift $\mathscr{X}$ should be something which depends only on the syntactic semigroup $\mathscr{X}$, or equivalently, on the action of $S(\mathscr{X})$ on the states of the Krieger cover. Formally, we say that an equivalence relation ϑ on sofic shifts is a *syntactic invariant* if $S(\mathscr{X}) \cong S(\mathscr{Y})$ implies that $\mathscr{X} \ \vartheta \ \mathscr{Y}$. Our final main result, proved in Section 13, states that the Karoubi envelope is the universal syntactic invariant of flow equivalence of sofic shifts in the sense that any other invariant factors through it.

Theorem 4.6. *Suppose that ϑ is a syntactic invariant of flow equivalence of sofic shifts and that $\mathscr{X}$ and $\mathscr{Y}$ are sofic shifts such that $\mathbb{K}(\mathscr{X})$ is equivalent to $\mathbb{K}(\mathscr{Y})$. Then $\mathscr{X} \ \vartheta \ \mathscr{Y}$.*

We remark that the sofic hypothesis is essential in Theorem 4.6. The expected generalization to all shifts does not hold. For example, if $\mathscr{X}$ is the union of a periodic shift $\mathscr{Y}$ with a minimal non-periodic shift, then $\mathbb{K}(\mathscr{X})$ and $\mathbb{K}(\mathscr{Y})$ are isomorphic, but $S(\mathscr{X})$ is infinite and $S(\mathscr{Y})$ is finite. And of course, finiteness of the syntactic semigroup is a flow invariant.

5. The proper comunication graph

For a graph G, let $PC(G)$ be the set of non-trivial strongly connected components of G. Here we consider a strongly connected graph to be trivial if it consists of one vertex and no edges (a single vertex with some loop edges is deemed non-trivial.) Consider in $PC(G)$ the partial order given by $C_1 \leq C_2$ if and only if there is in G a path from an element of C_1 to an element of C_2. Following the terminology of [2], the *proper communication graph of* G is the acyclic directed graph with vertex set $PC(G)$ and edge set given by the irreflexive relation $<$. It is proved in [2] that the proper communication graph of the right (left) Krieger cover of a sofic shift is a flow equivalence invariant. We shall see in this section that this result can naturally be seen as a consequence of Theorem 4.3, and in the process of doing this, we generalize it to arbitrary shifts.

Let S be a semigroup with zero acting on a pointed set Q. Let $I = Q \cdot E(S)$ and define a preorder $\preceq$ on I by $q \preceq q'$ if $q'LU(S) \subseteq qLU(S)$, that is, $q' = qs$ for some $s \in LU(S)$. As usual, define $q \sim q'$ if $q \preceq q'$ and $q' \preceq q$. Then $P(Q) = I/{\sim}$ is a poset, which can be identified with the poset of cyclic $LU(S)$-subsets $\{q \cdot LU(S) \mid q \in I\})$ ordered by reverse inclusion. In [15], we show that $P(Q)$ is a Karoubi invariant of the action of S on Q. More precisely, we have the following result that will appear in [15].

Theorem 5.1. *Suppose S, T are semigroups with zero acting on pointed sets Q, Q', respectively. Suppose that $F\colon \mathbb{K}(S) \longrightarrow \mathbb{K}(T)$ is an equivalence and that $\eta\colon \mathbb{A}_Q \Rightarrow A_{Q'} \circ F$ is an isomorphism. Then there is a well defined isomorphism $f\colon P(Q) \longrightarrow P(Q')$ of posets given by $f(q) = \eta_e(q)LU(T)$ for $q \in Qe$.*

Define for a shift $\mathscr{X}$, the poset $P(\mathscr{X}) = P(Q(\mathscr{X}))$. Note that $q \preceq q'$ if and only if there is a path from q to q' in $\mathfrak{K}^0(\mathscr{X})$.

Theorems 4.3 and 5.1 immediately imply the following.

Corollary 5.2. *The poset $P(\mathscr{X})$ is a flow equivalence invariant.*

Remark 5.3. If $\mathscr{X}$ is a synchronizing shift then, by Proposition 3.8, the elements of $Q_{\mathfrak{F}}(\mathscr{X})$ are the sink vertex $\emptyset$ and the vertices of the form $R_{\mathscr{X}}(u)$, with u a synchronizing word. Therefore, Proposition 3.4 implies that $qLU(\mathscr{X})$ belongs to $P(\mathscr{X})$ for every $q \in Q_{\mathfrak{F}}(\mathscr{X})$. Moreover, by Remark 4.4, one has $qLU(\mathscr{X}) = q'LU(\mathscr{X})$ for all $q, q' \in Q_{\mathfrak{F}}(\mathscr{X}) \setminus \{\emptyset\}$ and this common element is the minimum element of $P(\mathscr{X}) \setminus \{\emptyset\}$. Therefore, it follows from Theorem 5.1 that $q \in Q_{\mathfrak{F}}(\mathscr{X})$ if and only if $\eta_e(q) \in Q_{\mathfrak{F}}(\mathscr{Y})$, for every e with $q \in Qe$.

The following simple fact will be useful to specialize to sofic shifts.

Lemma 5.4. *Suppose that $\mathscr{X}$ is a sofic shift. Then q belongs to a non-trivial strongly connected component of $\mathfrak{K}^0(\mathscr{X})$ if and only if there is an idempotent $e \in S(\mathscr{X})$ such that $q = q \cdot e$.*

Proof. If $qe = q$ with $e \in E(S(\mathscr{X}))$, then any word representing e labels a loop at q and so q belongs to a non-trivial strongly connected component. Conversely, if q belongs to a non-trivial strongly connected component, then there is a word u labeling a non-empty loop rooted at q. Then $q = q \cdot [u^n]$ for all $n \geq 1$. On the other hand, since $S(\mathscr{X})$ is finite, there is $m \geq 1$ such that $[u^m]$ is idempotent (cf. [57, Proposition 1.6].) This completes the proof. $\square$

It is clear from Lemma 5.4 that $P(\mathscr{X}) \cong PC(\mathfrak{K}^0(\mathscr{X}))$ for a sofic shift $\mathscr{X}$.

Corollary 5.5. *The proper communication graph of the right (left) Krieger cover of a sofic shift is a flow equivalence invariant.*

6. The labeled preordered set of the $\mathcal{D}$-classes of $S(\mathscr{X})$

In this section we use Theorem 4.3 and a result from [15] to obtain a flow equivalence invariant (Theorem 6.2) which, as we shall observe, improves some related results from [6, 12]. We close the section with some examples.

6.1. Abstract semigroup setting.

Given a semigroup S, let $\mathfrak{D}(S)$ be the set of $\mathcal{D}$-classes of S. Endow $\mathfrak{D}(S)$ with the preorder $\preceq$ such that, if D_1 and D_2 belong to $\mathfrak{D}(S)$, then $D_1 \preceq D_2$ if and only if there are $d_1 \in D_1$ and $d_2 \in D_2$ such that d_2 is a factor of d_1. Note that if $\mathcal{D} = \mathcal{J}$ (for example, if S is finite), then $\preceq$ is a partial order.

If we assign to each element x of a preordered set P a label $\lambda_P(x)$ from some set, we obtain a new structure, called *labeled preordered set.* A morphism in the category of labeled preordered sets is a morphism $\varphi\colon P \longrightarrow Q$ of preordered sets such $\lambda_Q \circ \varphi = \lambda_P$.

Suppose that the semigroup S acts on a set Q. Then each element of S can be viewed as a transformation on Q. The *rank* of a transformation is the cardinality of its image. Elements of S which are $\mathcal{D}$-equivalent have the same rank, as transformations of Q (cf. [44, Proposition 4.1].) We define a labeled preordered set $\mathfrak{D}_\ell(S)$ as follows. The underlying preordered set is the set of $\mathcal{D}$-classes of $LU(S)$. The label of a $\mathcal{D}$-class D is (ε, H, r) where $\varepsilon \in \{0, 1\}$, with $\varepsilon = 1$ if and only if D is regular, H is the Schützenberger group of D and r is the rank of an element of D, viewed as a mapping on Q via the action of S. Denote by $\mathfrak{D}_Q(S)$ the resulting labeled poset. The following result will be proved in [15].

Theorem 6.1. *If S is a semigroup with zero acting on a pointed set Q, then $\mathfrak{D}_Q(S)$ is a Karoubi invariant of (Q, S).*

6.2. Application to shifts.

Consider a shift $\mathscr{X}$. Recall that $S(\mathscr{X})$ acts faithfully as a transformation semigroup on the set $Q(\mathscr{X})$, and that if $\mathscr{X}$ is synchronizing then the restriction of this action to $Q_{\mathfrak{F}}(\mathscr{X})$ is also faithful.

The sink state $\emptyset$ is not a vertex of the Krieger cover of $\mathscr{X}$. Therefore, following analogous conventions found in [6, 12], we let $\mathfrak{K}\mathfrak{D}(\mathscr{X})$ be the labeled preordered set obtained from $\mathfrak{D}_{Q(\mathscr{X})}(LU(\mathscr{X}))$, by replacing the label (ε, H, r) of each $\mathcal{D}$-class D by the label $(\varepsilon, H, r-1)$.

Similarly, if $\mathscr{X}$ is synchronizing, then we replace in $\mathfrak{D}_{Q_{\mathfrak{F}}(\mathscr{X})}(LU(\mathscr{X}))$ the label (ε, H, r) of each $\mathcal{D}$-class D by $(\varepsilon, H, r-1)$, and denote the resulting labeled preordered set by $\mathfrak{F}\mathfrak{D}(\mathscr{X})$. An immediate consequence of Theorems 4.3 and 6.1 is then the following.

Theorem 6.2. *For every shift $\mathscr{X}$, the labeled preordered set $\mathfrak{K}\mathfrak{D}(\mathscr{X})$ is a flow equivalence invariant. If $\mathscr{X}$ is synchronizing then $\mathfrak{F}\mathfrak{D}(\mathscr{X})$ is also a flow equivalence invariant.*

Theorem 6.2 improves on a weaker conjugacy invariant introduced in the doctoral thesis of the first author [14]. The invariance under conjugacy of $\mathfrak{K}\mathfrak{D}(\mathscr{X})$ when $\mathscr{X}$ is sofic (and of $\mathfrak{F}\mathfrak{D}(\mathscr{X})$ if $\mathscr{X}$ is moreover irreducible) was first proved in [12]. Forgetting the non-regular $\mathcal{D}$-classes, we extract the weaker conjugacy invariants of sofic shifts, first obtained in [6].

6.3. **Examples.**

Example 6.3. Let $\mathscr{X}$ and $\mathscr{Y}$ be the irreducible sofic shifts on the 14-letter alphabet $\{a_1, \ldots, a_{14}\}$ whose right Fischer covers are respectively represented in Figure 1. The dashed edges are those whose label appears only in one edge. The labeled ordered sets $\mathfrak{K}\mathfrak{D}(\mathscr{X})$ and $\mathfrak{K}\mathfrak{D}(\mathscr{Y})$ are represented in Fig-

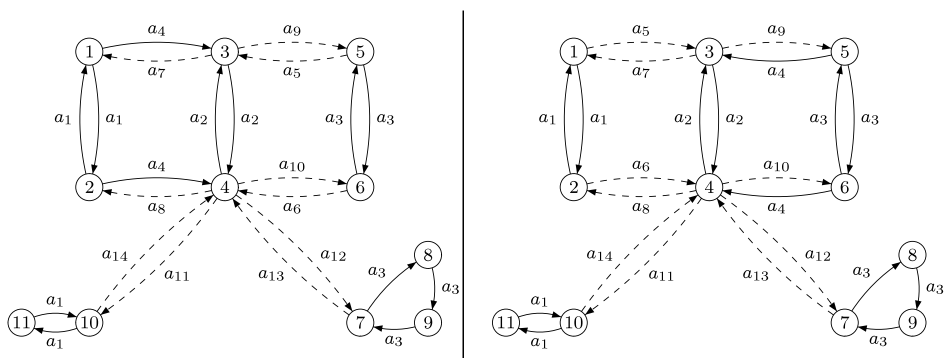

FIGURE 1. The irreducible sofic shifts $\mathscr{X}$ and $\mathscr{Y}$ in Example 6.3.

ure 2 by their Hasse diagrams, where within each node of the diagram we find information identifying the $\mathcal{D}$-class and its label. For example, in the first diagram, the notation $a_2|(1, C_2, 3)$ means the node represents the $\mathcal{D}$-class of $[a_2]_{\mathscr{X}}$ and that its label is the triple $(1, C_2, 3)$, where, as usual, C_n denotes the cyclic group of order n. The computations were carried out using GAP [18,19,26]. The labeled ordered sets $\mathfrak{K}\mathfrak{D}(\mathscr{X})$ and $\mathfrak{K}\mathfrak{D}(\mathscr{Y})$ are not isomorphic, hence $\mathscr{X}$ and $\mathscr{Y}$ are not flow equivalent.

Remark 6.4. In Example 6.3, the Fischer covers of $\mathscr{X}$ and $\mathscr{Y}$ have the same underlying labeled graph, as seen in Figure 1. The left Fischer covers, and the right and left Krieger covers of $\mathscr{X}$ and $\mathscr{Y}$, also have respectively the same underlying labeled graph. The interest in this remark is that Rune Johansen announced in his doctoral thesis [31] a result of Boyle, Carlsen and Eilers [10] implying that if two sofic shifts are flow equivalent then the edge shifts defined by the underlying unlabeled graphs of their right (left) Krieger covers are also flow equivalent, in a canonical way (cf. [31, Theorem 2.11 and Proposition 2.12]), and in the case of irreducible sofic shifts, the same happens with the edge shifts defined by the right (left) Fischer covers.

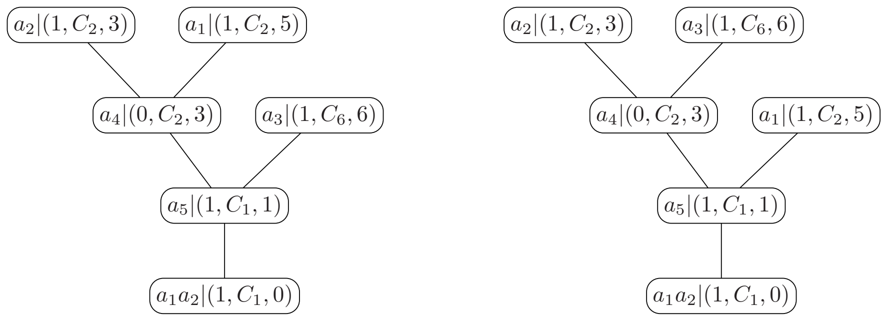

FIGURE 2. $\mathfrak{KD}(\mathscr{X})$ and $\mathfrak{KD}(\mathscr{Y})$.

Example 6.5. Let $\mathscr{X}$ and $\mathscr{Y}$ be the irreducible sofic shifts whose Fischer covers are respectively presented in Figure 3. Note that a^2 fixes all vertices,

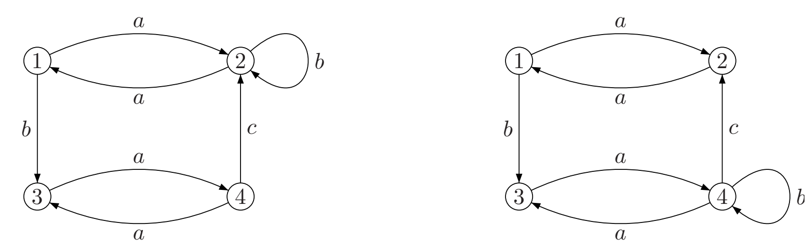

FIGURE 3. Two irreducible shifts $\mathscr{X}$ and $\mathscr{Y}$.

whence $S(\mathscr{X})$ and $S(\mathscr{Y})$ are monoids. The $\mathcal{D}$-class of the identity is at the top of the Hasse diagram of the $\mathcal{D}$-classes, and so, by Theorem 6.2, flow equivalent shifts whose syntactic semigroup is a monoid have the same number of vertices in the Krieger cover, and also in the Fischer cover if they are synchronizing. Using GAP [18, 26], one can check that the right Krieger covers of $\mathscr{X}$ and $\mathscr{Y}$ have the same proper communication graph, but distinct number of vertices (7 and 6, respectively.) Hence, $\mathfrak{KD}\big(S(\mathscr{X})\big) \not\simeq \mathfrak{KD}\big(S(\mathscr{Y})\big)$, and so $\mathscr{X}$ and $\mathscr{Y}$ are not flow equivalent by Theorem 6.2.

Clearly, two equivalent categories have the same local monoids, up to isomorphism. Therefore, from Theorem 4.2, we immediately extract the following criterion for sofic shifts whose syntactic semigroup is a monoid.

Proposition 6.6. *Let $\mathscr{X}$ and $\mathscr{Y}$ be flow equivalent shifts. If $S(\mathscr{X})$ is a monoid then $S(\mathscr{X})$ embeds into $S(\mathscr{Y})$. In particular, if $S(\mathscr{X})$ and $S(\mathscr{Y})$ are both finite monoids, then $S(\mathscr{X})$ and $S(\mathscr{Y})$ are isomorphic.* □

Example 6.7. We return to the shifts from Example 6.5. Simple computations with GAP show that the monoids $S(\mathscr{X})$ and $S(\mathscr{Y})$ are not isomorphic and so Proposition 6.6 again shows that $\mathscr{X}$ and $\mathscr{Y}$ are not flow equivalent.

Next we give an example concerning non-sofic synchronizing shifts.

Example 6.8. In Figure 4 one sees the Fischer covers of two sofic shifts, respectively $\mathscr{X}$ and $\mathscr{Y}$, both having bc as a synchronizing word (for checking that these are Fischer covers, one may use [21, Theorem 2.16].) If $u \in L(\mathscr{X})$,

FIGURE 4. Fischer covers of two synchronizing shifts.

then the rank of the action of $[u]_{\mathscr{X}}$ in $\mathfrak{F}(\mathscr{X})$ is one or infinite, depending on whether c is a factor of u or not. On the other hand, the idempotent $[av]_{\mathscr{Y}}$ has rank two. Therefore $\mathscr{X}$ and $\mathscr{Y}$ are not flow equivalent, by Theorem 6.2.

7. Classes of sofic shifts defined by classes of semigroups

Given a class V of semigroups with zero closed under isomorphism, we denote by $\mathcal{S}(\mathsf{V})$ the class of shifts $\mathscr{X}$ such that $S(\mathscr{X})$ belongs to V. The *localization of* V, denoted by $\mathcal{L}\mathsf{V}$, is the class of semigroups S whose local monoids eSe (with $e \in E(S)$) belong to V. We mention that it follows easily from known results that an irreducible shift $\mathscr{X}$ is of finite type if and only if $S(\mathscr{X}) \in \mathcal{L}\mathsf{Sl}$ (cf. [13, Proposition 4.2]) where Sl is the class of finite idempotent and commutative monoids.

Since the local monoids of S are precisely the endomorphism monoids of the category $\mathbb{K}(S)$, the following is immediate.

Theorem 7.1. *If V is a class of monoids with zero closed under isomorphism, then the class $\mathcal{S}(\mathcal{L}\mathsf{V})$ is closed under flow equivalence.*

More generally, one can use Theorem 4.2 to show that the classes proved in [13] to be closed under conjugacy are actually closed under flow equivalence.

Theorem 7.1 provides a method to show that some natural classes of sofic shifts are closed under flow equivalence. As an example, let us consider the class of *almost finite type* shifts. For background and motivation see [46, Section 13.1]. The shifts in Examples 6.3 and 6.5 are almost finite type shifts. Denote by $\mathcal{S}_I(\mathsf{V})$ the intersection of $\mathcal{S}(\mathsf{V})$ with the class of irreducible shifts. It turns out that the class of almost finite type shifts is $S_I(\mathcal{L}\,\mathsf{ECom})$, where ECom is the class of finite monoids with commuting idempotents [13]. Hence, from Theorem 7.1 we deduce the next result, first proved in [24].

Theorem 7.2. *The class of almost finite type shifts is closed under flow equivalence.* □

In [3, 6] a class of sofic shifts called *aperiodic shifts* is studied. In [6] it is proved that the class of irreducible aperiodic shifts is $S_I(\mathsf{A})$, where A is the class of finite *aperiodic semigroups* (a semigroup is *aperiodic* if the group of units of each of its local monoids is trivial), and that this class is closed under conjugacy. Since $\mathsf{A} = \mathcal{L}\mathsf{A}$ we obtain the following improvement.

Theorem 7.3. *The class of irreducible aperiodic shifts is closed under flow equivalence.* □

8. Markov-Dyck and Markov-Motzkin shifts

The examples we have given so far were mainly of sofic or synchronizing shifts. In this section we apply Theorem 4.2 to classify with respect to flow equivalence two classes of shifts, introduced and studied by W. Krieger and his collaborators [27, 38, 40, 41, 43], that in general are non-sofic and non-synchronizing. These shifts, called Markov-Dyck and Markov-Motzkin shifts, are the main subject of the article [43]. Previously, they had appeared as special cases of more general constructions in [27, 38, 40, 41]. These shifts are defined in terms of graph inverse semigroups.

An inverse semigroup S is a semigroup such that for all $s \in S$, there is a unique element s^* such that both $ss^*s = s$ and $s^*ss^* = s^*$ hold. Note that ss^*, s^*s are then both idempotents. Idempotents of an inverse semigroup commute and hence form a subsemigroup which is a semilattice. In an inverse semigroup, one has that $s \mathrel{\mathcal{L}} t$ (respectively, $s \mathrel{\mathcal{R}} t$) if and only if $ss^* = tt^*$ (respectively, $s^*s = t^*t$.) Consequently, one has $ss^* \mathrel{\mathcal{D}} s^*s$. See [54, 56] for more on inverse semigroups.

Graph inverse semigroups, in the special case of graphs with no multiple-edges, were first considered by Ash and Hall [1]. They were considered in greater generality in [54, 55]. The recent paper [32] is a useful reference.

Let G be a (directed) graph and let G^* be the free category generated by G. The objects of G^* are the vertices of G and the morphisms are (directed) paths, including an empty path at each vertex. Since we are adopting the Category Theory convention for the composition of morphisms, in this context a path of G should be understood as follows: a non-empty finite sequence of edges $x_1 \cdots x_n$ is a path of G if the domain of x_i is the range of x_{i+1}, for $1 \leq i < n$. Of course, the empty path at a vertex q, denoted 1_q, is the identity of G^* at q. The domain and range of a path u will be denoted respectively by αu and ωu.

Associated to G is an inverse semigroup P_G, called the *graph inverse semigroup of* G. The underlying set of P_G is the set of all pairs (x, y) of morphisms $x, y \in G^*$ such that x and y have a common domain, together with an extra element 0, which is the zero element of P_G. The pair (x, y) is usually denoted xy^{-1}. We think on y^{-1} as a formal inverse of the path y, obtained by reversing the directions of the edges in y. We make the identifications $y = (y, 1_{\alpha y})$ and $y^{-1} = (1_{\alpha y}, y)$. Moreover, we have $1_q^{-1} = 1_q$, for every vertex q of G. The multiplication between elements of $P_G \setminus \{0\}$ is given by the following rules:

$$xy^{-1} \cdot uv^{-1} = \begin{cases} xzv^{-1} & \text{if } u = yz \text{ for some path } z, \\ x(vz)^{-1} & \text{if } y = uz \text{ for some path } z, \\ 0 & \text{otherwise.} \end{cases}$$

The semigroup P_G is an inverse semigroup in which $(xy^{-1})^* = yx^{-1}$ and $0^* = 0$. Its non-zero idempotents are the pairs of the form xx^{-1}, with $x \in G^*$. Note that 1_q is idempotent, for every vertex q of G, and $x^{-1}x = 1_{\alpha x}$ for every $x \in G^*$.

Let S be an inverse semigroup with zero and let Σ be a set of (semigroup) generators for S. Let $\pi\colon \Sigma^* \longrightarrow S$ be the canonical projection. Then $L = \pi^{-1}(S \setminus \{0\})$ is a factorial and prolongable language and hence we have an associated shift (cf. [46, Proposition 1.3.4].)

From now on, we assume that the graph G is finite. Let Σ_G be the set of elements of P_G of the form x or x^{-1}, with x an edge of G. In general, P_G is generated by the union Λ_G of Σ_G and the set of empty paths 1_q. Let $\rho\colon (\Lambda_G)_0^+ \longrightarrow P_G$ be the canonical projection. The *Markov-Motzkin shift* M_G is the shift with language $\rho^{-1}(P_G \setminus \{0\})$.

If the out-degree of each vertex is at least 1, then P_G is generated by Σ_G because if q is a vertex and x is an out-going edge from q, then $1_q = x^{-1}x$. We may therefore consider the unique onto homomorphism $\pi\colon (\Sigma_G)_0^+ \longrightarrow P_G$ extending the identity map. The *Markov-Dyck shift* D_G is the shift with language $L(D_G) = \pi^{-1}(P_G \setminus \{0\})$.

Note that the syntactic semigroups of D_G and M_G are homomorphic images of P_G, as P_G recognizes $L(D_G)$ and $L(M_G)$. Let us now characterize when P_G is the syntactic semigroup of M_G and D_G. An inverse semigroup S is called *fundamental*, or an *antigroup*, if the only congruence contained in the $\mathcal{H}$-relation is the trivial one.

Lemma 8.1. *Let S be an inverse semigroup with zero. Then S is 0-disjunctive if and only if S is fundamental and $E(S)$ is 0-disjunctive.*

Lemma 8.1 is Lemma IV.3.10 from [56]. It is instrumental in the following proof of the desired characterization of M_G and P_G.

Lemma 8.2. *Let G be a finite graph. Then P_G is the syntactic semigroup of M_G if and only if G has no vertex of in-degree exactly one. If, in addition, each vertex has out-degree at least one, then P_G is the syntactic semigroup of D_G if and only if G has no vertex of in-degree exactly one.*

Proof. In view of Lemma 3.2, it suffices to check when P_G is 0-disjunctive. By Lemma 8.1, this occurs if and only if P_G is fundamental and $E(P_G)$ is 0-disjunctive. It turns out that the $\mathcal{H}$-relation in P_G is trivial [32, Theorem 2.1], and so P_G is fundamental. On the other hand, it is shown in [32, Lemma 2.20] that $E(P_G)$ is 0-disjunctive[2] if and only if G has no vertex of in-degree exactly one. $\square$

It is easy to see that P_G is finite if and only if G is acyclic. Therefore, the above lemma implies that the shifts M_G and D_G are not sofic in general.

[2]The definition of 0-disjunctive semilattice given in [32, Lemma 2.20] appears in a different form, which is however equivalent to the one we are using, see [56, Exercise IV.3.13(ii)].

In view of our mains results and of Lemma 8.2, we are naturally interested in investigating the Karoubi envelope of P_G. It is for that purpose, that we introduce the next notions. Recall that a morphism $f\colon d \longrightarrow c$ of a category C is said to be a *split monomorphism* if it has a left inverse, that is, if there is a morphism $g\colon c \longrightarrow d$ such that $gf = 1_d$. The composition of two split monomorphisms is a split monomorphism, and so we can consider the subcategory L_C of C formed by the split monomorphisms of C. Note that an equivalence $C \longrightarrow D$ restricts to an equivalence $L_C \longrightarrow L_D$.

For a semigroup S, a morphism (e, s, f) of $\mathbb{K}(S)$ is a split monomorphism if and only if $s\ \mathcal{L}\ f$ [48, Proposition 3.1]. In particular, if S has a zero, then the only split monomorphisms (e, s, f) with $s = 0$ are of the form $(e, 0, 0)$. In other words, if we consider the full subcategory $\mathbb{L}(S)$ of $L_{\mathbb{K}(S)}$ whose objects are the non-zero idempotents, then there are no morphisms of the form $(e, 0, f)$ in this subcategory. Note that an equivalence $L_{\mathbb{K}(S)} \longrightarrow L_{\mathbb{K}(T)}$ restricts to an equivalence $\mathbb{L}(S) \longrightarrow \mathbb{L}(T)$ because 0 is the unique initial object of $L_{\mathbb{K}(S)}$ and similarly for $L_{\mathbb{K}(T)}$.

The argument for the following lemma is essentially contained in [59].

Lemma 8.3. *For every graph G, the category $\mathbb{L}(P_G)$ is equivalent to the free category G^*.*

Proof. An object of $\mathbb{L}(P_G)$ is an idempotent of the form uu^{-1}, with $u \in G^*$. Since $uu^{-1}\ \mathcal{D}\ 1_{\alpha u}$, the category $\mathbb{L}(P_G)$ is equivalent to the full subcategory $\mathbb{L}(P_G)'$ whose objects are the idempotents of the form 1_q. If $u, v \in G^*$ are such that $(1_q, uv^{-1}, 1_r)$ is a split monomorphism of $\mathbb{K}(P_G)$, then $1_r = vu^{-1}uv^{-1} = vv^{-1}$ and so $v = 1_r$. Therefore, $\mathbb{L}(P_G)'$ is isomorphic to G^*. □

It is well known that two free categories G^* and H^* on graphs G, H are equivalent if and only if G and H are isomorphic. Since we don't know a precise reference, we sketch a proof.

Lemma 8.4. *Suppose that G, H are graphs with G^* equivalent to H^*. Then G and H are isomorphic.*

Proof. In a free category, there are no isomorphisms other than the identities. Hence, two functors with codomain a free category are isomorphic if and only if they are equal. It follows that any equivalence $F\colon G^* \longrightarrow H^*$ is actually an isomorphism. A morphism u of a free category is an edge if and only if it cannot be factored $u = vw$ with v, w non-empty paths. Thus F must restrict to a graph isomorphism $G \longrightarrow H$. □

As a corollary of the preceding two lemmas, we deduce that Morita equivalent graph inverse semigroups have isomorphic underlying graphs.

Corollary 8.5. *Let G, H be graphs. Then P_G is Morita equivalent to P_H if and only if G and H are isomorphic.*

The case where G is a finite one-vertex graph with at least 2 loops has received special attention in the literature and motivates the general case.

If $N \geq 2$ is the number of loops of the one-vertex graph G, then the corresponding Markov-Dyck shift is denoted D_N, and is called the *Dyck shift of rank N*. In [50] it is proved, by means of the computation of certain flow invariant abelian groups, that if D_N and D_M are flow equivalent then $N = M$. This is proved here again, as a special case of the next result.

Theorem 8.6. *Let G and H be finite graphs such that the in-degree of none of their vertices is exactly one. Then the Markov-Motzkin shifts M_G and M_H are flow equivalent if and only if G and H are isomorphic. If in addition, the out-degree of each vertex of G and H is at least one, then the Markov-Dyck shifts D_G and D_H are flow equivalent if and only if G and H are isomorphic.*

Proof. Sufficiency is obvious. Conversely, suppose that M_G and M_H are flow equivalent. By Lemma 8.2, P_G and P_H are the syntactic semigroups of M_G and M_H, respectively. They are inverse semigroups and hence have local units. Theorem 4.2 implies that P_G and P_H are Morita equivalent. Corollary 8.5 then yields $G \cong H$. The proof for Markov-Dyck shifts is identical. $\square$

Theorem 8.6 generalizes results of Krieger [40, 41], who showed that if G and H are strongly connected and each vertex of G and H has in-degree at least 2, then D_G is conjugate to D_H if and only if G and H are isomorphic. Krieger's proof relies on the invariance of a semigroup that is studied in the next section.

9. Shifts with Krieger's Property ($\mathcal{A}$)

Wolfgang Krieger introduced in [40] (see also [28, 29, 42]) the class of shifts with property ($\mathcal{A}$) whose definition will be recalled in this section. To each shift with property ($\mathcal{A}$), Krieger associated a semigroup which he showed to be a conjugacy invariant. Recently, Krieger announced that property ($\mathcal{A}$) is a flow equivalence invariant as is the corresponding semigroup. In this section, we give an alternative characterization of the Krieger semigroup in terms of the Karoubi envelope, from which we deduce a new proof of its flow invariance, under a condition usually assumed when studying property ($\mathcal{A}$).

9.1. **Non-zero divisors.** Let C be a small category. Two morphisms f and g of C are *isomorphic*, denoted $f \cong g$, when there are isomorphisms φ and ψ of C such that $f = \varphi g \psi$. The relation $\cong$ is an equivalence relation on the set of morphisms of C satisfying the property that if $f \cong g$ and $f' \cong g'$, with the compositions ff' and gg' defined, then $ff' \cong gg'$.

Suppose moreover that C is a category with zero. We may then consider the equivalence relation $\cong_0$ refining $\cong$ by identifying all zero morphisms into a single partition class, leaving intact the $\cong$-classes of non-zero morphisms. Denote by $C/{\cong_0}$ the set of equivalence classes defined by $\cong_0$. We use the notation $\langle f \rangle$ for the $\cong_0$-equivalence class of f. In the absence of ambiguity, the $\cong_0$-class of the zero morphisms is denoted 0.

A category in which each of its hom-sets has at most one element is essentially the same thing as a preorder [47] and so we call them preorders.

Definition 9.1. Consider a small category C with zero. Let T be a subcategory of C which is a preorder and contains the non-zero isomorphisms of C. We define a binary operation $\circ_T$ on $C/{\cong_0}$ as follows (note that the operation is well defined because T contains the non-zero isomorphisms):

(1) For every x in $C/{\cong_0}$ we have $0 \circ_T x = x \circ_T 0 = 0$.
(2) Suppose f_1 and f_2 are non-zero morphisms of C. If there is a (necessarily unique) morphism $h\colon r(f_2) \longrightarrow d(f_1)$ in T, then $\langle f_1 \rangle \circ_T \langle f_2 \rangle = \langle f_1 h f_2 \rangle$, otherwise $\langle f_1 \rangle \circ_T \langle f_2 \rangle = 0$.

The operation $\circ_T$ endows $C/{\cong_0}$ with a structure of semigroup with zero. We denote by $C_{\circ_T}$ this semigroup.

For a small category C with zero, a non-zero morphism f of C is called a *non-zero divisor* if $gf \neq 0$ and $fh \neq 0$ whenever g, h are non-zero morphisms with $d(g) = r(f)$ and $d(f) = r(h)$. The non-zero divisors form a subcategory C_{nzd} of C containing all non-zero isomorphisms of C. If $T = C_{\mathsf{nzd}}$ is a preorder, then we denote the operation $\circ_T$ by $\bullet$, and the semigroup $C_{\circ_T}$ by $C_\bullet$. The proof of the following proposition is a routine exercise.

Proposition 9.2. *Let C and D be equivalent small categories with zero. Then C_{nzd} is equivalent to D_{nzd}. In particular, C_{nzd} is a preorder if and only if D_{nzd} is a preorder. If C_{nzd} and D_{nzd} are preorders, then the semigroups $C_\bullet$ and $D_\bullet$ are isomorphic.*

Let S be a semigroup with zero. Let us say that a non-zero morphism (e, s, f) of $\mathbb{K}(S)$ is a *strong non-zero divisor* if we have $rs \neq 0$ and $st \neq 0$ whenever r and t are elements of S such that $re \neq 0$ and $ft \neq 0$. The strong non-zero divisors of $\mathbb{K}(S)$ form a subcategory $T = \mathbb{K}(S)_{\mathsf{snzd}}$. If T is a preorder, then we denote $\circ_T$ by $\circ$, and $\mathbb{K}(S)_{\circ_T}$ by $\mathbb{K}(S)_\circ$. We shall see in this section that the semigroup associated by Krieger to a shift $\mathscr{X}$ with property $(\mathcal{A})$ is $\mathbb{K}(\mathscr{X})_\circ$ (Theorem 9.19.)

Note that a strong-non zero divisor morphism is a non-zero divisor. In the next proposition, whose easy proof we omit, a sufficient condition for the converse is given.

Proposition 9.3. *Suppose S is a semigroup with zero S satisfying the following condition: for every $s \in S \setminus \{0\}$, there are $r, t \in S$ and idempotents $e, f \in S$ such that $erstf \neq 0$. Then we have $\mathbb{K}(S)_{\mathsf{snzd}} = \mathbb{K}(S)_{\mathsf{nzd}}$.*

Remark 9.4. Suppose that $\mathscr{X}$ is a sofic subshift of $A^{\mathbb{Z}}$. All sufficiently long words of A^+ have a factor that maps to an idempotent in $S(\mathscr{X})$ (cf. [57, Theorem 1.11]), from which it follows that $S(\mathscr{X})$ satisfies the condition in Proposition 9.3, thus $\mathbb{K}(\mathscr{X})_{\mathsf{snzd}} = \mathbb{K}(\mathscr{Y})_{\mathsf{nzd}}$.

9.2. **The set $\mathcal{A}(\mathscr{X})$.** In this section, $\mathscr{X}$ is always a subshift of $A^{\mathbb{Z}}$. For each $u \in A^+$, let

$$\omega^+(u) = \{v \in L(\mathscr{X}) \mid wu \in L(\mathscr{X}) \Rightarrow wuv \in L(\mathscr{X})\},$$
$$\omega^-(u) = \{v \in L(\mathscr{X}) \mid uw \in L(\mathscr{X}) \Rightarrow vuw \in L(\mathscr{X})\}.$$

Consider also the following subsets of $\mathscr{X}$:

$$\mathcal{A}_n^+(\mathscr{X}) = \{x \in \mathscr{X} \mid \forall i \in \mathbb{Z},\, x_i \in \omega^+(x_{[i-n,i-1]})\},$$
$$\mathcal{A}_n^-(\mathscr{X}) = \{x \in \mathscr{X} \mid \forall i \in \mathbb{Z},\, x_i \in \omega^-(x_{[i+1,i+n]})\}.$$

Remark 9.5. Suppose that $uv \in L(\mathcal{A}_n^+(\mathscr{X}))$ and that u has length at least n. Then $wu \in L(\mathscr{X})$ implies $wuv \in L(\mathscr{X})$. A dual remark holds for $\mathcal{A}_n^-(\mathscr{X})$.

If non-empty, the intersection $\mathcal{A}_n(\mathscr{X}) = \mathcal{A}_n^+(\mathscr{X}) \cap \mathcal{A}_n^-(\mathscr{X})$ is an n-step finite type subshift of $A^{\mathbb{Z}}$. Note that $n \leq m$ implies $\mathcal{A}_n(\mathscr{X}) \subseteq \mathcal{A}_m(\mathscr{X})$. Finally, denote by $\mathcal{A}(\mathscr{X})$ the union $\bigcup_{n\geq 1} \mathcal{A}_n(\mathscr{X})$.

Remark 9.6. If $\mathscr{X}$ is an n-step finite shift, then $\mathscr{X} = \mathcal{A}_n(\mathscr{X}) = \mathcal{A}(\mathscr{X})$.

From hereon, we denote by $\mathscr{E}(\mathscr{X})$ the set of words of A^+ such that $[u]$ is a non-zero idempotent. The length of an element $w \in A^*$ is denoted $|w|$. W. Krieger, in his talk at the workshop *Flow equivalence of graphs, shifts and C^*-algebras*, held at the University of Copenhagen in November 2013 indicated that the following lemma should hold.

Lemma 9.7. *If $u \in \mathscr{E}(\mathscr{X})$ then the periodic point u^∞ belongs to $\mathcal{A}_{2|u|}(\mathscr{X})$.*

Proof. The fact that $[u]$ is a non-zero idempotent guarantees $u^\infty \in \mathscr{X}$. Let $a \in A$ and $v \in A^+$ be such that $|v| = 2|u|$ and va is a factor of u^∞. We want to show that $a \in \omega^+(v)$. Let $p \in A^+$ be such that $pv \in L(\mathscr{X})$. Since $|v| = 2|u|$ and va is a factor of u^∞, there is a word z conjugate to u for which $v = z^2$ and the letter a is a prefix of z, whence of v. Lemma 2.4 implies that $[v]$ is idempotent, and so from $pv \in L(\mathscr{X})$ we get $pv^2 \in L(\mathscr{X})$, thus $pva \in L(\mathscr{X})$. This shows that $a \in \omega^+(v)$, establishing $u^\infty \in \mathcal{A}^+_{2|u|}(\mathscr{X})$. Dually, we have $u^\infty \in \mathcal{A}^-_{2|u|}(\mathscr{X})$. □

Two elements x and y of $A^{\mathbb{Z}}$ are *left asymptotic* if there is $n \in \mathbb{Z}$ such that $x_i = y_i$ for all $i \leq n$. Of course, there is the dual notion of *right asymptotic*. If x is left asymptotic to p^∞ and right asymptotic to q^∞, then there is $u \in A^+$ such that x is in the orbit of the bi-infinite sequence $\ldots ppp.uqqq\ldots$. We represent by $p^{-\infty}uq^{+\infty}$ this sequence. In absence of confusion, we may represent also by $p^{-\infty}uq^{+\infty}$ any sequence in its orbit.

For a subset Z of $\mathscr{X}$, let $P(Z)$ be the set of periodic points of $\mathscr{X}$ belonging to Z. We define a relation $\preceq$ in $P(\mathcal{A}(\mathscr{X}))$ by letting $q \preceq r$ if there is $z \in \mathcal{A}(\mathscr{X})$ such that z is left asymptotic to q and z is right asymptotic to r. Since $\mathcal{A}_n(\mathscr{X})$ is a finite type shift for every n, the relation $\preceq$ is transitive, thus a preorder. The resulting equivalence is denoted $\sim$.

Proposition 9.8. *Let $p,q \in \mathscr{E}(\mathscr{X})$ and $u \in A^+$. Then $p^{-\infty}uq^{+\infty}$ belongs to $\mathcal{A}(\mathscr{X})$ if and only if the morphism $([p],[puq],[q])$ of $\mathbb{K}(\mathscr{X})$ is a strong non-zero divisor.*

Proof. Suppose $n \geq 1$ is such that $p^{-\infty}uq^{+\infty} \in \mathcal{A}_n(\mathscr{X})$. Let $w \in A^+$ with $[wp] \neq 0$. As $[p]$ is idempotent, we have $wp^n \in L(\mathscr{X})$. Therefore, since $p^{-\infty}uq^{+\infty} \in \mathcal{A}_n(\mathscr{X})$, we have $wp^nu \in L(\mathscr{X})$ by Remark 9.5. Again because $[p]$ is idempotent, we conclude that $[wpuq] \neq 0$. Dually, if $[qw] \neq 0$ then $[puqw] \neq 0$. Hence, $([p],[puq],[q])$ is a strong non-zero divisor.

Conversely, suppose that $([p],[puq],[q])$ is a strong non-zero divisor. By Lemma 9.7, there is $n \geq 1$ with $p^\infty, q^\infty \in \mathcal{A}_n(\mathscr{X})$. Let $m = \max\{n, |puq|\}$. Consider a factor of $p^{-\infty}uq^{+\infty}$ of the form za, with $a \in A$, and $|z| = m$. We claim that $a \in \omega^+(z)$. That holds if za is a factor of p^∞ or q^∞, as $|z| \geq n$ and $p^\infty, q^\infty \in \mathcal{A}_n(\mathscr{X})$. So, suppose that za is neither a factor of p^∞, nor of q^∞. Then, since $|z| \geq |puq|$, there are only two possibilities (see Figure 5):

(1) there are $k \geq 1$ and $\alpha, \beta \in A^*$ such that $z = \alpha p\beta$ and $\alpha puq^k \in zaA^*$;
(2) there are $k \geq 2$ and $\alpha, \beta \in A^*$ such that $z = \alpha q\beta$ and $\alpha q^k \in zaA^*$.

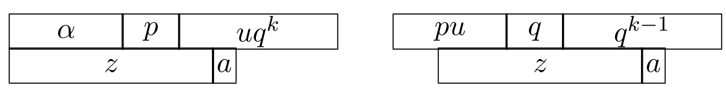

FIGURE 5. The two cases where za is not a factor of p^∞ or of q^∞.

Assume we are in case (1), and let $x \in A^*$ be such that $xz \in L(\mathscr{X})$. Then, as $z = \alpha p\beta$, we have $[x\alpha p] \neq 0$. Therefore, $[x\alpha puq^k] \neq 0$ because $([p],[puq],[q])$ is a strong non-zero divisor morphism of $\mathbb{K}(\mathscr{X})$. And since $\alpha puq^k \in zaA^*$, we conclude that $xza \in L(\mathscr{X})$, thus $a \in \omega^+(z)$.

Suppose we are in case (2), and let $x \in A^*$ with $xz \in L(\mathscr{X})$. As $z = \alpha q\beta$ and $[q]$ is idempotent, we have $x\alpha q^k\beta \in L(\mathscr{X})$. Since $\alpha q^k \in zaA^*$, it follows that $xza \in L(\mathscr{X})$, thus $a \in \omega^+(z)$.

All cases exhausted, we have shown that $p^{-\infty}uq^{+\infty} \in \mathcal{A}^+_m(\mathscr{X})$. Symmetrically, $p^{-\infty}uq^{+\infty} \in \mathcal{A}^-_m(\mathscr{X})$ holds. Hence, we have $p^{-\infty}uq^{+\infty} \in \mathcal{A}_m(\mathscr{X})$. $\square$

Corollary 9.9. *Let $p,q \in \mathscr{E}(\mathscr{X})$ be such that $[p]\ \mathcal{D}\ [q]$. Then $p^\infty \sim q^\infty$. More precisely, if $u,v \in A^+$ are such that $[p] = [uv]$ and $[q] = [vu]$ then $p^{-\infty}uq^{+\infty} \in \mathcal{A}(\mathscr{X})$ and $q^{-\infty}vp^{+\infty} \in \mathcal{A}(\mathscr{X})$.*

Proof. By Lemma 2.3, as $[p]$ and $[q]$ are idempotents, $[p]\ \mathcal{D}\ [q]$ means there are $u,v \in A^+$ such that $[p] = [uv]$ and $[q] = [vu]$. Moreover, $([p],[puq],[q])$ and $([q],[qvp],[p])$ are isomorphisms, whence strong non-zero divisors. $\square$

9.3. **Property ($\mathcal{A}$).**

Definition 9.10 (Property ($\mathcal{A}$))**.** Let $n \geq 1$ and $H \geq 0$ be integers. The shift $\mathscr{X}$ is said to have *property* $(\mathcal{A}, n, H)$ when the following conditions hold: if p,u,v,q are elements of A^* such that $puq, pvq \in L(\mathcal{A}_n(\mathscr{X}))$ and $|p| = |q| = H$, then $[puq]_{\mathscr{X}} = [pvq]_{\mathscr{X}}$. The shift $\mathscr{X}$ has *property ($\mathcal{A}$)* if, for every $n \geq 1$, there is $H \geq 0$ such that $\mathscr{X}$ has property $(\mathcal{A}, n, H)$.

For shifts with property $(\mathcal{A})$, we have a sort of converse of Lemma 9.7, also suggested by W. Krieger.

Lemma 9.11. *Suppose that $\mathscr{X}$ has property $(\mathcal{A})$. If $u^\infty \in \mathcal{A}(\mathscr{X})$, then there is $h_0 \geq 1$ such that $u^h \in \mathscr{E}(\mathscr{X})$ for every $h \geq h_0$.*

Proof. Let $n \geq 1$ be such that $u^\infty \in \mathcal{A}_n(\mathscr{X})$, and let H be such that $\mathscr{X}$ has property $(\mathcal{A}, n, H)$. Take $h_0 = 2H + 1$. Let $h = h_0 + d$, where d is a nonnegative integer. Then, by the definition of property $(\mathcal{A})$, the words $u^h = u^H \cdot u^{d+1} \cdot u^H$ and $u^{2h} = u^H \cdot u^{2H+2d+2} \cdot u^H$ of $L(\mathcal{A}_n(\mathscr{X}))$ satisfy $[u^h] = [u^{2h}]$, that is, $[u^h]$ is idempotent. □

From Lemmas 9.7 and 9.11, we immediately deduce the following characterization of periodic points of $\mathcal{A}(\mathscr{X})$.

Corollary 9.12. *If $\mathscr{X}$ has property $(\mathcal{A})$, then the periodic points of $\mathcal{A}(\mathscr{X})$ are the elements of $\mathscr{X}$ of the form p^∞ with $p \in \mathscr{E}(\mathscr{X})$.*

It follows from Corollary 9.12 that if a sofic shift $\mathscr{X}$ has property $(\mathcal{A})$, then $P(\mathcal{A}(\mathscr{X}))$ is dense in $\mathscr{X}$ so long as $\mathscr{X}$ is non-wandering (meaning that for $u \in L(\mathscr{X})$, there exists $v \in A^*$ such that $uvu \in L(\mathscr{X})$.) When studying shifts with property $(\mathcal{A})$, it is sometimes assumed in the first place that $\mathcal{A}(\mathscr{X})$ is dense in $\mathscr{X}$ [28, 29].

For a shift $\mathscr{X}$ with property $(\mathcal{A})$, isomorphism between objects of $\mathbb{K}(\mathscr{X})$ has a transparent dynamical description in terms of the equivalence relation $\sim$ between periodic points of $\mathcal{A}(\mathscr{X})$.

Proposition 9.13. *Suppose that $\mathscr{X}$ has property $(\mathcal{A})$. There is a well-defined bijection $\Lambda\colon P(\mathcal{A}(\mathscr{X}))/{\sim} \longrightarrow (E(S(\mathscr{X}))\setminus\{0\})/\mathcal{D}$ given by $\Lambda(p^\infty/{\sim}) = [p]/\mathcal{D}$ for $p \in \mathscr{E}(\mathscr{X})$.*

Proof. To show that Λ is well defined, recall first from Lemma 9.11 that every element of $P(\mathcal{A}(\mathscr{X}))$ is of the form p^∞ for some $p \in \mathscr{E}(\mathscr{X})$.

Now, let $p, q \in \mathscr{E}(\mathscr{X})$ with $p^\infty/{\sim} = q^\infty/{\sim}$. Then there are $n \geq 1$ and $u, v \in A^+$ such that $p^{-\infty}uq^{+\infty}$ and $q^{-\infty}vp^{+\infty}$ belong to $\mathcal{A}_n(\mathscr{X})$. Let H be such that $\mathscr{X}$ has property $(\mathcal{A}, n, H)$. Since $\mathcal{A}_n(\mathscr{X})$ is an n-step finite type shift, we have $p^H uq^n vp^H \in L(\mathcal{A}_n(\mathscr{X}))$. Therefore, by the definition of property $(\mathcal{A})$, we have $[p^{2H+1}] = [p^H uq^n vp^H]$. Since $[p]$ and $[q]$ are idempotents, we get $[p] = [puqvp]$, and so $[p] \mathrel{\mathcal{R}} [puq]$. Similarly, we have $[q] = [qvpuq]$ and $[q] \mathrel{\mathcal{L}} [puq]$. We deduce $[p] \mathrel{\mathcal{D}} [q]$, showing that Λ is well defined.

That Λ is injective follows immediately from Corollary 9.9. Surjectivity is a direct consequence of Lemma 9.7. □

We proceed in the task of relating property $(\mathcal{A})$ with properties of $\mathbb{K}(\mathscr{X})$.

Proposition 9.14. *If a shift $\mathscr{X}$ has property $(\mathcal{A})$ then $\mathbb{K}(\mathscr{X})_{\mathsf{snzd}}$ is a preorder. Conversely, if $\mathscr{X}$ is sofic and $\mathbb{K}(\mathscr{X})_{\mathsf{snzd}}$ is a preorder, then $\mathscr{X}$ has property $(\mathcal{A})$.*

Proof. Suppose that $([p],[u],[q])$ and $([p],[v],[q])$ are morphisms of $\mathbb{K}(\mathscr{X})_{\mathsf{snzd}}$. By Proposition 9.8, there is $n \geq 1$ such that $p^{-\infty}uq^{+\infty}$ and $p^{-\infty}vq^{+\infty}$ belong to $\mathcal{A}_n(\mathscr{X})$. Let H be such that $\mathscr{X}$ has property $(\mathcal{A}, n, H)$. Since $p^H uq^H$ and $p^H vq^H$ belong to $L(\mathcal{A}_n(\mathscr{X}))$, we have $[p^H uq^H] = [p^H vq^H]$ by the definition of property $(\mathcal{A}, n, H)$, thus $[u] = [v]$, as $[u] = [puq]$ and $[v] = [pvq]$. Therefore, $\mathbb{K}(\mathscr{X})_{\mathsf{snzd}}$ is a preorder.

Conversely, suppose that $\mathscr{X}$ is a sofic shift for which $\mathbb{K}(\mathscr{X})_{\mathsf{snzd}}$ is a preorder. Fix $n \geq 1$. It is well known that if φ is a homomorphism from A^+ onto a finite semigroup S, then there is an integer N_φ such that every word v with length at least N_φ has a factor w with $|w| \geq n$ and $\varphi(w)$ idempotent (cf. [57, Theorem 1.11].) Take $H = N_{\delta_{\mathscr{X}}}$. Let $puq \in L(\mathcal{A}_n(\mathscr{X}))$ be such that $|p|, |q| \geq H$. Then there are factorizations $p = p'\alpha p''$ and $q = q'\beta q''$, with $|\alpha|, |\beta| \geq n$ and $[\alpha], [\beta]$ idempotents. We claim that the morphism $([\alpha], [\alpha p'' uq' \beta], [\beta])$ of $\mathbb{K}(\mathscr{X})$ is a strong non-zero divisor. Let $z \in A^+$ be such that $z\alpha \in L(\mathscr{X})$. Note that $\alpha p'' uq' \beta$ is an element of $\mathcal{A}_n(\mathscr{X})$, as it is a factor of puq. Hence, as $|\alpha| \geq n$, we have $z\alpha p'' uq' \beta \in L(\mathscr{X})$ by Remark 9.5. Therefore, $[z\alpha] \neq 0$ implies $[z\alpha p'' uq' \beta] \neq 0$, and, dually, $[\beta z] \neq 0$ implies $[\alpha p'' uq' \beta z] \neq 0$, proving the claim that $([\alpha], [\alpha p'' uq' \beta], [\beta])$ is a strong non-zero divisor. Moreover, if $pvq \in L(\mathcal{A}_n(\mathscr{X}))$, then $([\alpha], [\alpha p'' vq' \beta], [\beta])$ is also a strong non-zero divisor. By the hypothesis that $\mathbb{K}(\mathscr{X})_{\mathsf{snzd}}$ is a preorder, it follows that $[\alpha p'' uq' \beta] = [\alpha p'' vq' \beta]$, whence $[puq] = [pvq]$. This shows that $\mathscr{X}$ has property $(\mathcal{A}, n, H)$. □

Proposition 9.14 gives an effective decision process for property $(\mathcal{A})$ for sofic shifts, solving a problem posed by W. Krieger at the workshop *Flow equivalence of graphs, shifts and C^*-algebras*, held at the University of Copenhagen in November 2013. The related problem raised by W. Krieger at that workshop as to whether there exist sofic shifts without property $(\mathcal{A})$ is solved in the positive, as follows from the next corollary.

Corollary 9.15. *If $\mathscr{X}$ is a shift with property $(\mathcal{A})$, then $S(\mathscr{X})$ is aperiodic.*

In particular, the even shift [46] consisting of all bi-infinite words in $\{0,1\}^{\mathbb{Z}}$ containing no factor of the form $10^{2n+1}1$ with $n \geq 0$ is a sofic shift without property $(\mathcal{A})$.

Proof of Corollary 9.15. Every automorphism of $\mathbb{K}(\mathscr{X})$ is a strong non-zero divisor. Since $\mathbb{K}(\mathscr{X})_{\mathsf{snzd}}$ is a preorder, it follows from Proposition 9.14 that the automorphism groups of $\mathbb{K}(\mathscr{X})$ are trivial. By Lemma 2.3, this means that the maximal subgroups of $S(\mathscr{X})$ are trivial, i.e., $S(\mathscr{X})$ is aperiodic. □

9.4. **The Krieger semigroup.** Let $\mathscr{X}$ be a shift with property $(\mathcal{A})$. Denote by $Y(\mathscr{X})$ the set of elements of $\mathscr{X}$ of the form $p^{-\infty}uq^{+\infty}$, with $p, u, q \in A^+$ and $p^\infty, q^\infty \in \mathcal{A}(\mathscr{X})$. By Corollary 9.12, we can assume that $p, q \in \mathscr{E}(\mathscr{X})$. The following definition is from [40], and appears also in [28, 29, 42].

Definition 9.16. For each integer $n \geq 1$, suppose that $\mathscr{X}$ has property $(\mathcal{A}, n, H)$. Consider on $Y(\mathscr{X})$ the relation $\approx_{n,H}$ defined as follows. Two elements $x_1, x_2 \in Y(\mathscr{X})$ satisfy $x_1 \approx_{n,H} x_2$ when there are words p_i, q_i, u_i such that $x_i = p_i^{-\infty} u_i q_i^{+\infty}$, with $p_i^{\infty}, q_i^{\infty} \in \mathcal{A}_n(\mathscr{X})$, for $i = 1, 2$, and the following conditions are satisfied:

(1) $p_1^{\infty} \sim p_2^{\infty}$, $q_1^{\infty} \sim q_2^{\infty}$;
(2) $[z_1] = [z_2]$ whenever z_1 and z_2 are words for which there are factorizations

$$z_1 = \pi\zeta_1 v_1 u_1 w_1 \xi_1 \rho, \qquad z_2 = \pi\zeta_2 v_2 u_2 w_2 \xi_2 \rho, \tag{9.1}$$

with:
(a) $|\pi| = |\rho| = |v_i| = |w_i| = H$;
(b) $p_i^{-\infty}$ ends with v_i and $q_i^{+\infty}$ begins with w_i;
(c) the words $\pi\zeta_i v_i$ and $w_i \xi_i \rho$ belong to $L(\mathcal{A}_n(\mathscr{X}))$.

We write $x_1 \approx x_2$ if and only if $x_1 \approx_{n,H} x_2$ for some n, H such that $\mathscr{X}$ has property $(\mathcal{A}, n, H)$.

Krieger noted that the relation $\approx$ is a well-defined equivalence relation.

Proposition 9.17. *Let $\mathscr{X}$ be a shift with property ($\mathcal{A}$). Let p_i, u_i, q_i be words such that $p_i^{-\infty} u_i q_i^{+\infty} \in Y(\mathscr{X})$ and $p_i, q_i \in \mathscr{E}(\mathscr{X})$, for $i = 1, 2$. Then $p_1^{-\infty} u_1 q_1^{+\infty} \approx p_2^{-\infty} u_2 q_2^{+\infty}$ if and only if the morphisms $([p_1], [p_1u_1q_1], [q_1])$ and $([p_2], [p_2u_2q_2], [q_2])$ of $\mathbb{K}(\mathscr{X})$ are isomorphic.*

Proof. Suppose the morphisms $([p_1], [p_1u_1q_1], [q_1])$ and $([p_2], [p_2u_2q_2], [q_2])$ are isomorphic. Then, there are words $\alpha, \beta, \gamma, \delta$ such that

$$[p_1] = [\alpha\beta], \quad [p_2] = [\beta\alpha], \qquad \text{and} \qquad [q_1] = [\gamma\delta], \quad [q_2] = [\delta\gamma], \tag{9.2}$$

and for which the following diagram in $\mathbb{K}(\mathscr{X})$ commutes:

$$\begin{tikzcd} {[p_1]} & & {[q_1]} \arrow[ll, "{([p_1],[p_1u_1q_1],[q_1])}"'] \arrow[d, "{([q_2],[q_2\delta q_1],[q_1])}"] \\ {[p_2]} \arrow[u, "{([p_1],[p_1\alpha p_2],[p_2])}"] & & {[q_2]}. \arrow[ll, "{([p_2],[p_2u_2q_2],[q_2])}"'] \end{tikzcd} \tag{9.3}$$

Hence, we have

$$[p_1u_1q_1] = [p_1\alpha p_2 u_2 q_2 \delta q_1]. \tag{9.4}$$

By Lemma 9.7 and Corollary 9.9, there is $n \geq 1$ such that the sequences p_i^{∞}, q_i^{∞} (for $i = 1, 2$), $p_1^{-\infty}\alpha p_2^{+\infty}$ and $q_2^{-\infty}\delta q_1^{+\infty}$ belong to $\mathcal{A}_n(\mathscr{X})$. Let h be such that $\mathscr{X}$ has property $(\mathcal{A}, n, h)$. As we can replace p_i by p_i^{nh}, and q_i by q_i^{nh}, without changing the elements of $\mathscr{X}$ which have appeared so far, we may as well assume that the words p_i, q_i have length at least nh.

Take $H = |p_1||q_1||p_2||q_2|$. We claim that $p_1^{-\infty} u_1 q_1^{+\infty} \approx_{n,H} p_2^{-\infty} u_2 q_2^{+\infty}$. Let z_1, z_2 be words with factorizations as in (9.1), satisfying conditions (2a)-(2c) in Definition 9.16. Since $p_1^{\infty} \sim p_2^{\infty}$ and $q_1^{\infty} \sim q_2^{\infty}$ by Corollary 9.9, to prove the claim it remains to show that $[z_1] = [z_2]$. As $H = |v_i|$ is a multiple of $|p_i|$ and $p_i^{-\infty}$ ends with v_i, we have $v_i = p_i^{\frac{H}{|p_i|}}$, for $i = 1, 2$. Similarly,

$w_i = q_i^{\frac{H}{|q_i|}}$. Hence, $[v_i u_i w_i] = [p_i u_i q_i]$. Taking into account (9.1) and (9.4), we conclude that

$$[z_1] = [\pi\zeta_1 p_1 \alpha p_2 u_2 q_2 \delta q_1 \xi_1 \rho]. \tag{9.5}$$

Recall that $\pi\zeta_i v_i$ belongs to $L(\mathcal{A}_n(\mathscr{X}))$, whence so does its prefix $\pi\zeta_i p_i$. Then, since $p_1^{-\infty}\alpha p_2^{+\infty} \in \mathcal{A}_n(\mathscr{X})$, $|p_1| \geq n$ and $\mathcal{A}_n(\mathscr{X})$ is an n-step finite type shift, we deduce that $\pi\zeta_1 p_1 \alpha p_2 \in L(\mathcal{A}_n(\mathscr{X}))$. As $|\pi| = |p_2| = H \geq h$, and $\mathscr{X}$ has property $(\mathcal{A}, n, h)$, we then obtain $[\pi\zeta_1 p_1 \alpha p_2] = [\pi\zeta_2 p_2]$. Similarly, we have $[q_2 \delta q_1 \xi_1 \rho] = [q_2 \xi_2 \rho]$. Hence, in (9.5) we can replace $\pi\zeta_1 p_1 \alpha p_2$ and $q_2 \delta q_1 \xi_1 \rho$ respectively by $\pi\zeta_2 p_2$ and $q_2 \xi_2 \rho$, deducing $[z_1] = [z_2]$ via (9.1).

Conversely, suppose that $p_1^{-\infty} u_1 q_1^{+\infty} \approx p_2^{-\infty} u_2 q_2^{+\infty}$ holds. Then we have $p_1^{\infty} \sim p_2^{\infty}$ and $q_1^{\infty} \sim q_2^{\infty}$, thus $[p_1]\, \mathcal{D}\, [p_2]$ and $[q_1]\, \mathcal{D}\, [q_2]$, by Proposition 9.13. We may therefore consider words $\alpha, \beta, \gamma, \delta$ as in (9.2). By Corollary 9.9, there is n such that $p_1^{-\infty}\alpha p_2^{+\infty}$ and $q_2^{-\infty}\delta q_1^{+\infty}$ belong to $\mathcal{A}_n(\mathscr{X})$. Suppose $\mathscr{X}$ has property $(\mathcal{A}, n, H)$. Let $z_1 = p_1^{2H} u_1 q_1^{2H}$ and $z_2 = p_1^H \alpha p_2^H u_2 q_2^H \delta q_1^H$. Then clearly z_1 and z_2 admit factorizations of the form (9.1), satisfying conditions (2a)-(2c). Therefore, applying the hypothesis $p_1^{-\infty} u_1 q_1^{+\infty} \approx p_2^{-\infty} u_2 q_2^{+\infty}$, we obtain $[z_1] = [z_2]$. This means that (9.4) holds, as $[p_i], [q_i]$ are idempotents. That is, (9.3) commutes, which shows that $([p_1], [p_1 u_1 q_1], [q_1])$ and $([p_2], [p_2 u_2 q_2], [q_2])$ are isomorphic. $\square$

We next describe an operation on $Y(\mathscr{X})/{\approx}$, first introduced by Krieger in [40], and later reprised in [28, 29, 42], its presentation showing small variations between these papers.

Definition 9.18. Consider a shift $\mathscr{X}$ with property $(\mathcal{A})$. Let $x, y \in Y(\mathscr{X})$. Suppose the following sequence of conditions holds:

(1) $x = p^{-\infty} u \alpha^{+\infty}$ and $y = \beta^{-\infty} v q^{+\infty}$ for some words $u, v \in A^+$ and $p, q, \alpha, \beta \in \mathscr{E}(\mathscr{X})$ (cf. Corollary 9.12);
(2) $\mathscr{X}$ has property $(\mathcal{A}, n, H)$ for some n, H;
(3) $\alpha^{-\infty} v \beta^{+\infty} \in \mathcal{A}_n(\mathscr{X})$ for some word v;
(4) there is $z \in \mathscr{X}$ such that $z = p^{-\infty} u \alpha' w \beta' v q^{+\infty}$, for some words w, α', β' such that $|\alpha'|, |\beta'| \geq H$, the sequence $\alpha^{+\infty}$ begins with α', the sequence $\beta^{-\infty}$ ends with β', and $\alpha' w \beta' \in L(\mathcal{A}_n(\mathscr{X}))$.

Then one defines $[x]_\approx \cdot [y]_\approx = [z]_\approx$. If it is not possible to establish such a sequence of conditions, then one defines $[x]_\approx \cdot [y]_\approx = 0$, where 0 is an extra element.

Krieger observed that the binary operation appearing in Definition 9.18 defines the structure of a semigroup $(Y(\mathscr{X})/{\approx})^0$ with zero, whose underlying set is $Y(\mathscr{X})/{\approx}$ together with the extra element 0. We call this semigroup the *Krieger semigroup of* $\mathscr{X}$. If the shift $\mathscr{X}$ has property $(\mathcal{A})$, then, thanks to Proposition 9.14, we may consider the semigroup $\mathbb{K}(\mathscr{X})_\circ$. The following is the main result of this section.

Theorem 9.19. *Let $\mathscr{X}$ be a shift with property ($\mathcal{A}$). The Krieger semigroup of $\mathscr{X}$ is isomorphic to $\mathbb{K}(\mathscr{X})_\circ$.*

Proof. For a shift $\mathscr{X}$ with property ($\mathcal{A}$), let us denote by τ the bijection from $(Y(\mathscr{X})/{\approx})^0$ to $\mathbb{K}(\mathscr{X})_\circ$ defined by $\tau(0) = 0$ and $\tau(p^{-\infty}uq^{+\infty}) = \big\langle([p],[u],[q])\big\rangle$, when $p^{-\infty}uq^{+\infty} \in Y(\mathscr{X})$ with $p,q \in \mathscr{E}(\mathscr{X})$. That τ is well defined and injective, follows from Proposition 9.17, and that it is surjective, follows from Lemma 9.7.

Let $x, y \in Y(\mathscr{X})$. The proof of the theorem is concluded once we establish the equality

$$\tau([x]_\approx \cdot [y]_\approx) = \tau([x]_\approx) \circ \tau([y]_\approx). \tag{9.6}$$

Suppose that $\tau([x]_\approx) \circ \tau([y]_\approx) \neq 0$. We may take words $p, \alpha, \beta \in \mathscr{E}(\mathscr{X})$ and $u, v \in A^+$ such that $x = p^{-\infty}u\alpha^{+\infty}$, $y = \beta^{-\infty}vq^{+\infty}$ and $[u] = [pu\alpha]$, $[v] = [\beta vq]$. Then $\tau([x]_\approx) = \big\langle([p],[u],[\alpha])\big\rangle$ and $\tau([y]_\approx) = \big\langle([\beta],[v],[q])\big\rangle$. The hypothesis $\tau([x]_\approx) \circ \tau([y]_\approx) \neq 0$ implies the existence of a strong non-zero divisor of the form $([\alpha],[w],[\beta])$ such that

$$\tau([x]_\approx) \circ \tau([y]_\approx) = \big\langle([p],[uwv],[q])\big\rangle \tag{9.7}$$

and $[uwv] \neq 0$. In particular, for every $k \geq 1$, we have

$$p^{-\infty}u\alpha^k w\beta^k vq^{+\infty} \in \mathscr{X} \tag{9.8}$$

and

$$\tau(p^{-\infty}u\alpha^k w\beta^k vq^{+\infty}) = \big\langle([p],[uwv],[q])\big\rangle. \tag{9.9}$$

Moreover, as $([\alpha],[w],[\beta])$ is a strong non-zero divisor morphism, we know by Proposition 9.8 that there is $n \geq 1$ such that

$$\alpha^k w\beta^k \in L(\mathcal{A}_n(\mathscr{X})), \tag{9.10}$$

for every $k \geq 1$, thus $\alpha^{-\infty}w\beta^{+\infty} \in \mathcal{A}_n(\mathscr{X})$. If we choose $k \geq 1$ such that $\mathscr{X}$ has property $(\mathcal{A}, n, k)$, we deduce from (9.8) and (9.10) that we are in the conditions of Definition 9.18 (by taking $\alpha' = \alpha^k$, $\beta' = \beta^k$) in such a way that the following equality holds:

$$[x]_\approx \cdot [y]_\approx = [p^{-\infty}u\alpha^k w\beta^k vq^{+\infty}]_\approx.$$

This, together with (9.9) and (9.7), establishes (9.6).

Conversely, suppose that $[x]_\approx \cdot [y]_\approx \neq 0$. Take the same notational setting as in Definition 9.18. Note that $\tau([z]_\approx) = \big\langle([p],[pu\alpha' w\beta' vq],[q])\big\rangle$. Since the words $\alpha^H v\beta^H$ and $\alpha' w\beta'$ belong to $L(\mathcal{A}_n(\mathscr{X}))$, have the same prefix of length H and the same suffix of length H, and since $\mathscr{X}$ has property $(\mathcal{A}, n, H)$, we have $[\alpha' w\beta'] = [\alpha^H v\beta^H]$. As $[\alpha]$ and $[\beta]$ are idempotents, we actually have $[\alpha' w\beta'] = [\alpha v\beta]$, and so the following holds:

$$\tau([z]_\approx) = \big\langle([p],[pu\alpha],[\alpha])([\alpha],[\alpha v\beta],[\beta])([\beta],[\beta vq],[q])\big\rangle. \tag{9.11}$$

By Proposition 9.8, the morphism $([\alpha],[\alpha v\beta],[\beta])$ is a strong non-zero divisor, and so we get (9.6).

Therefore, we proved that $[x]_\approx \cdot [y]_\approx \neq 0$ if and only if $\tau([x]_\approx) \circ \tau([y]_\approx) \neq 0$, and that (9.6) always holds. $\square$

Krieger announced in the aforementioned workshop in Copenhagen that the Krieger semigroup is flow invariant. We next give a proof of this fact via Theorem 9.19, under the usual assumption of density of $\mathcal{A}(\mathscr{X})$.

Proposition 9.20. *Let $\mathscr{X}$ be a shift with property ($\mathcal{A}$). Suppose that $\mathcal{A}(\mathscr{X})$ is dense in $\mathscr{X}$ or $\mathscr{X}$ is sofic. Then $\mathbb{K}(\mathscr{X})_{\mathsf{snzd}} = \mathbb{K}(\mathscr{X})_{\mathsf{nzd}}$, and so $\mathbb{K}(\mathscr{X})_\circ$ is invariant under flow equivalence.*

Proof. By Propositions 9.2 and 9.3, it suffices to show that $S(\mathscr{X})$ satisfies the condition in the statement of Proposition 9.3. This is the case if $\mathscr{X}$ is sofic, as seen in Remark 9.4. Assume that $\mathcal{A}(\mathscr{X})$ is dense. Take $u \in A^+$ with $[u]_{\mathscr{X}} \neq 0$. As $\mathcal{A}(\mathscr{X})$ is dense in $\mathscr{X}$, there is $n \geq 1$ and $x \in \mathcal{A}_n(\mathscr{X})$ such that u is a factor of x. Let n, H be such that $\mathscr{X}$ has property $(\mathcal{A}, n, H)$. Since $\mathcal{A}_n(\mathscr{X})$ is a finite type shift, there are words w_1, w_2, v_1, v_2 such that $[w_1 v_1 u v_2 w_2]_{\mathcal{A}_n(\mathscr{X})} \neq 0$ with $[w_1]_{\mathcal{A}_n(\mathscr{X})}$ and $[w_2]_{\mathcal{A}_n(\mathscr{X})}$ idempotents. As $[w_i]_{\mathcal{A}_n(\mathscr{X})} = [w_i^H]_{\mathcal{A}_n(\mathscr{X})} = [w_i^{2H}]_{\mathcal{A}_n(\mathscr{X})} \neq 0$ and $\mathscr{X}$ has property $(\mathcal{A}, n, H)$, we deduce that $e_i = [w_i^H]_{\mathscr{X}}$ is an idempotent of $S(\mathscr{X})$, for $i = 1, 2$. But $[w_1^H v_1 u v_2 w_2^H]_{\mathcal{A}_n(\mathscr{X})} \neq 0$, which implies $e_1 [v_1 u v_2]_{\mathscr{X}} e_2 \neq 0$, showing that $S(\mathscr{X})$ satisfies the hypothesis in Proposition 9.3. $\square$

10. Subsynchronizing subshifts of a sofic shift

As a further application of our main results, we apply them to the poset of subsynchronizing subshifts of a sofic shift considered in [34]. This poset, whose definition is recalled in this section, provides information about the structure of a reducible sofic shift.

Let $\mathscr{X}$ be a sofic subshift of $A^{\mathbb{Z}}$. We recall some definitions and remarks from [34]. If m is a synchronizing word for $\mathscr{X}$, then m is *magic* for $\mathscr{X}$ if $mum \in L(\mathscr{X})$ for some $u \in A^*$. If m is magic for $\mathscr{X}$, then the set

$$\{v \in A^+ \mid \exists x \in A^* : mxv \in L(\mathscr{X})\}$$

is the set of finite blocks of a sofic subshift of $\mathscr{X}$. This shift is denoted $S(m)$. If M is a set of magic words for $\mathscr{X}$, then $S(M)$ denotes the sofic shift $\bigcup_{m \in M} S(m)$. A subshift of $\mathscr{X}$ of the form $S(M)$ is called a *subsynchronizing subshift of $\mathscr{X}$*. The set $\mathsf{Subs}(\mathscr{X})$ of subsynchronizing subshifts of $\mathscr{X}$ is finite; see Lemma 10.2 below. It may be empty. If $\mathscr{X}$ is irreducible, then $\mathsf{Subs}(\mathscr{X}) = \{\mathscr{X}\}$.

Let $\mathscr{X}$ be a sofic shift. Say that $s \in S(\mathscr{X})$ is *synchronizing* if $s = [u]$ with u synchronizing. Note that s is synchronizing if and only if $rs, st \neq 0$ implies $rst \neq 0$, for all $r, t \in S(\mathscr{X})$. It follows from Lemma 3.3 that the synchronizing elements of $S(\mathscr{X})$, together with 0, form an ideal and that a synchronizing element s is idempotent if and only if $s^2 \neq 0$. A synchronizing idempotent of $S(\mathscr{X}) \setminus \{0\}$ will be called a *magic idempotent* for $\mathscr{X}$.

Remark 10.1. If $\mathscr{X}$ is a sofic shift, and $s \neq 0$, then there are $r, t \in S(\mathscr{X})$ and idempotents $g, h \in S(\mathscr{X})$ such that $grsth \neq 0$. Therefore, an idempotent e of $S(\mathscr{X})$ is magic if and only if the following happens: whenever (e, x, f) and

(g,y,e) are non-zero morphisms of $\mathbb{K}(\mathscr{X})$, the composition $(g,y,e)(e,x,f)$ is a non-zero morphism of $\mathbb{K}(\mathscr{X})$.

Let e be a magic idempotent for $\mathscr{X}$, and let u be a word such that $e=[u]$. Then u is a magic word for $\mathscr{X}$. Clearly, if $[u]=[v]$, then $S(u)=S(v)$. We may then define $S(e)$ as being $S(u)$. If M is a set of magic idempotents for $\mathscr{X}$, then $S(M)$ denotes the sofic shift $\bigcup_{e\in M} S(e)$.

In the proof of the following lemma, $\mathrm{Fac}(X)$ denotes the set of non-empty words which are factors of a language X.

Lemma 10.2. *If m is a magic word for $\mathscr{X}$, then $S(m)=S(e)$ for some magic idempotent e for $\mathscr{X}$.*

Proof. As m is a magic word for $\mathscr{X}$, there is $u\in L(\mathscr{X})$ with $mum\in L(\mathscr{X})$. We claim that $e=[mu]$ is a magic idempotent for $\mathscr{X}$ such that $S(m)=S(e)$.

We first show that e is an idempotent. It follows from Lemma 3.3 that we must show $(mu)^2\in L(\mathscr{X})$. But $mum, mu\in L(\mathscr{X})$ implies $mumu\in L(\mathscr{X})$. Also mu is synchronizing by Lemma 3.3. Thus e is a magic idempotent.

We have $L(S(m))=\mathrm{Fac}(R_{\mathscr{X}}(m))$ and $L(S(e))=\mathrm{Fac}(R_{\mathscr{X}}(mu))$ by definition. Clearly, $\mathrm{Fac}(R_{\mathscr{X}}(mum))\subseteq \mathrm{Fac}(R_{\mathscr{X}}(mu))\subseteq \mathrm{Fac}(R_{\mathscr{X}}(m))$. But $R_{\mathscr{X}}(mum)=R_{\mathscr{X}}(m)$ by Lemma 3.3. This shows that $L(S(m))=L(S(e))$, thus $S(m)=S(e)$. □

In the following lemma, we see how the inclusion relation between elements of $\mathsf{Subs}(\mathscr{X})$ is codified in the Karoubi envelope.

Lemma 10.3. *Let e be a magic idempotent for $\mathscr{X}$, and let M be a set of magic idempotents for $\mathscr{X}$. The following conditions are equivalent:*

1. *(1) $S(e)\subseteq S(f)$ for some $f\in M$;*
2. *(2) $S(e)\subseteq S(M)$;*
3. *(3) for some $f\in M$, there is a non-zero morphism $e\longrightarrow f$ in $\mathbb{K}(\mathscr{X})$.*

Proof. The implication (1)⇒(2) is trivial.

Let us show (2) ⇒(3). Assuming $S(e)\subseteq S(M)$, let $v\in L(\mathscr{X})$ be such that $e=[v]$. As $v\in S(e)$, by hypothesis there is $f\in M$ with $v\in S(f)$. Let $m\in L(\mathscr{X})$ be such that $f=[m]$. Then $mxv\in L(\mathscr{X})$ for some $x\in A^*$. Hence, $(f,[mxv],e)$ is a non-zero morphism $e\longrightarrow f$ in $\mathbb{K}(\mathscr{X})$.

Finally, suppose that $(f,[u],e)$ is a non-zero morphism $e\longrightarrow f$ in $\mathbb{K}(\mathscr{X})$ such that $f\in M$. Let $w\in S(e)$. If $m,v\in L(\mathscr{X})$ are synchronizing words such that $f=[m]$ and $e=[v]$, then $muv\in L(\mathscr{X})$ and $vxw\in L(\mathscr{X})$ for some $x\in A^*$. But then $muvxw\in L(\mathscr{X})$ by synchronization, and so $w\in S(f)$, thus showing that $S(e)\subseteq S(f)$ and establishing (1)⇒ (2). □

Corollary 10.4. *If e,f are magic idempotents for $\mathscr{X}$ such that $e\mathrel{\mathcal{D}} f$, then $S(e)=S(f)$.*

Proof. Since $e\mathrel{\mathcal{D}} f$, there are isomorphisms $e\longrightarrow f$ and $f\longrightarrow e$. As $e,f\neq 0$, these isomorphisms are non-zero, thus $S(e)=S(f)$ by Lemma 10.3. □

Remark 10.1 ensures that being a magic idempotent is a property preserved by equivalence of Karoubi envelopes, and so in the next result the set $S(F(M))$ is well defined.

Proposition 10.5. *Let $F\colon \mathbb{K}(\mathscr{X}) \longrightarrow \mathbb{K}(\mathscr{Y})$ be an equivalence, where $\mathscr{X}$ and $\mathscr{Y}$ are sofic shifts. Then the mapping $\Psi_F\colon \mathsf{Subs}(\mathscr{X}) \longrightarrow \mathsf{Subs}(\mathscr{Y})$ defined by $\Psi_F(S(M)) = S(F(M))$, where M runs over the sets of magic idempotents, is a well-defined isomorphism of posets. If G is a quasi-inverse of F, then Ψ_G is the inverse of Ψ_F.*

Proof. Let M and N be sets of magic idempotents with $S(M) \subseteq S(N)$. Since equivalences preserve non-zero morphisms, we have $S(F(M)) \subseteq S(F(N))$ by Lemma 10.3. This shows that Ψ_F is a well-defined and order-preserving function. Moreover, since $GF(e)\ \mathcal{D}\ e$ for every idempotent e, we conclude from Corollary 10.4 that Ψ_F and Ψ_G are mutually inverse. □

As a direct consequence of Theorem 4.2 and Proposition 10.5, we deduce the following.

Corollary 10.6. *The order structure of the poset of subsynchronizing subshifts of a sofic shift is invariant under flow equivalence.* □

The invariance of the order structure of $\mathsf{Subs}(\mathscr{X})$ under conjugacy of sofic shifts was proved in [34]. Concrete examples were examined in that paper. Actually a more general result was obtained in [34]: viewing $\mathsf{Subs}(\mathscr{X})$ as a labeled poset, where the label of each element is its conjugacy class, one obtains a conjugacy invariant. We shall not give a new proof of this fact using our methods, since we would not obtain a significative simplification. However, we do generalize it to flow equivalence, in the next theorem. For a sofic shift $\mathscr{X}$, its *labeled flow poset of subsynchronizing subshifts* is the labeled poset obtained from $\mathsf{Subs}(\mathscr{X})$ in which the label of each element of $\mathsf{Subs}(\mathscr{X})$ is its flow equivalence class.

Theorem 10.7. *The labeled flow poset of subsynchronizing subshifts of a sofic shift is a flow equivalence invariant.*

Proof. In view of the proof of Corollary 10.6 and the results from [34], it suffices to show that if $\mathscr{X}'$ is the symbol expansion of $\mathscr{X}$ relatively to a letter α, then there is an equivalence $F\colon \mathbb{K}(\mathscr{X}) \longrightarrow \mathbb{K}(\mathscr{X}')$ such that $S(F(e))$ is the symbol expansion of $S(e)$ relatively to α, whenever e is a magic idempotent for $\mathscr{X}$. This is done in an appendix, in Proposition A.1, using some tools introduced in Section 12. □

11. Eventual conjugacy

It remains a major open problem to determine whether conjugacy is decidable for shifts of finite type. To attack this problem, a relation called *eventual conjugacy*, also known as *shift equivalence*, was introduced, which may be defined as follows (see [46, Chapter 7] for historical background and other details.) Let n be a positive integer. Recall that A^n denotes the subset of A^+ of words with length n. Considering the natural embedding of $(A^n)^+$ into A^+, one defines the n^{th} *higher power* of a subshift $\mathscr{X}$ of $A^{\mathbb{Z}}$ as the subshift $\mathscr{X}^n$ of $(A^n)^{\mathbb{Z}}$ such that $L(\mathscr{X}^n) = L(\mathscr{X}) \cap (A^n)^+$. Two shifts $\mathscr{X}$ and $\mathscr{Y}$ are *eventually conjugate* if and only if $\mathscr{X}^n$ and $\mathscr{Y}^n$ are conjugate for all sufficiently large n (cf. [46, Definition 1.4.4].) Kim and Roush proved that eventual conjugacy for sofic shifts is decidable [35], but the algorithm which is available is quite intricate. Another deep result by Kim and Roush [36, 37] is that, for shifts of finite type, eventual conjugacy is not the same as conjugacy.

It is easy to check that, for every shift $\mathscr{X} \subseteq A^{\mathbb{Z}}$ and $u, v \in (A^n)^+$, we have $[u]_{\mathscr{X}^n} = [v]_{\mathscr{X}^n}$ if and only if $[u]_{\mathscr{X}} = [v]_{\mathscr{X}}$, and so $S(\mathscr{X}^n)$ embeds naturally in $S(\mathscr{X})$. For sofic shifts, we have the following sort of converse, taken from [12, Lemma 5.1] following arguments from [6].

Lemma 11.1. *Let $\mathscr{X}$ be a sofic shift. For each idempotent $e \in S(\mathscr{X})$, choose $u_e \in A^+$ such that $[u_e]_{\mathscr{X}} = e$. Let $A_{\mathscr{X}} = \prod_{e \in E(S(\mathscr{X}))} |u_e|$. Then, for every $n \geq 1$, we have $LU(\mathscr{X}) = LU(\mathscr{X}^{nA_{\mathscr{X}}+1})$.*

Let N be such that $\mathscr{X}^n$ and $\mathscr{Y}^n$ are conjugate for all $n \geq N$. Consider the integer $k = NA_{\mathscr{X}}A_{\mathscr{Y}}+1$. Then $LU(\mathscr{X}^k) = LU(\mathscr{X})$ and $LU(\mathscr{Y}^k) = LU(\mathscr{Y})$ by Lemma 11.1. Since $\mathscr{X}^k$ and $\mathscr{Y}^k$ are conjugate, this justifies the following result, implicitly used in [12].

Corollary 11.2. *If $\mathscr{X}$ and $\mathscr{Y}$ are eventually conjugate sofic shifts, then, for every integer N, there is $k \geq N$, with $\gcd(k, N) = 1$, and $\mathscr{X}^k$ and $\mathscr{Y}^k$ conjugate, such that $LU(\mathscr{X}^k) = LU(\mathscr{X})$ and $LU(\mathscr{Y}^k) = LU(\mathscr{Y})$.* □

Remark 11.3. For a labeled graph $\mathfrak{G}$, let $\mathfrak{G}^n$ be the labeled graph over A^n defined as follows: the vertices are those of $\mathfrak{G}$, and an edge from a vertex p to a vertex q, with label $u \in A^n$, is a path in $\mathfrak{G}$ from p to q with label u. It is easy to see that, up to isomorphism, the Krieger cover of $\mathscr{X}^n$ is the labeled graph $\mathfrak{K}(\mathscr{X})^n$, and, if $\mathscr{X}$ is synchronizing, the Fischer cover of $\mathscr{X}^n$ is $\mathfrak{F}(\mathscr{X})^n$. Moreover, the action of $S(\mathscr{X}^n)$ on $Q(\mathscr{X}^n)$ is the restriction to $S(\mathscr{X}^n)$ of the action of $S(\mathscr{X})$ on $Q(\mathscr{X})$.

Theorem 11.4. *Suppose that $\mathscr{X}$ and $\mathscr{Y}$ are eventually conjugate sofic shifts. Then the actions $\mathbb{A}_{\mathscr{X}}$ and $\mathbb{A}_{\mathscr{Y}}$ are equivalent. The same happens with the actions $\mathbb{A}^{\mathfrak{F}}_{\mathscr{X}}$ and $\mathbb{A}^{\mathfrak{F}}_{\mathscr{Y}}$ if $\mathscr{X}$ and $\mathscr{Y}$ are irreducible. In particular, the equivalence class of $\mathbb{K}(\mathscr{X})$ is an eventual conjugacy invariant of sofic shifts.*

Proof. By Corollary 11.2, there is k such that $\mathscr{X}^k$ is conjugate to $\mathscr{Y}^k$, $\mathbb{K}(\mathscr{X}^k) = \mathbb{K}(\mathscr{X})$ and $\mathbb{K}(\mathscr{Y}^k) = \mathbb{K}(\mathscr{Y})$. The result follows then from Theorems 4.3 and 4.5 in view of Remark 11.3. □

12. Proofs of Theorems 4.2, 4.3 and 4.5

This section is divided in three parts. First we establish the versions of Theorem 4.2 and Theorem 4.3 obtained by replacing "flow equivalence" by "conjugacy" and then the corresponding version for symbol expansion. In the third part we deduce Theorem 4.5.

12.1. **Invariance under conjugacy.** The proofs we present here of the conjugacy invariance part of Theorems 4.2 and 4.3 are via the results of Nasu [51]. A direct, but technical, proof appeared in the original version of this paper [17].

The conjugacy invariance part of Theorems 4.2 and 4.3 are immediate consequences of the "only if" part of Theorem 3.9 and of the following lemma (which should be compared to [4, Proposition 7.5].)

Lemma 12.1. *Suppose that $\mathcal{A}$ is a bipartite right-resolving labeled graph with components $\mathcal{A}_1, \mathcal{A}_2$ (with corresponding state sets Q_1, Q_2.) Let S_i be the transition semigroup of $\mathcal{A}_i$, for $i = 1, 2$, and let S be the transition semigroup of $\mathcal{A}$. Let S, S_1, S_2 act on Q^0, Q_1^0 and Q_2^0, respectively. Then the Karoubi envelopes of S, S_1 and S_2 are all equivalent and the actions $(\mathbb{A}_{Q^0}, \mathbb{K}(S))$, $(\mathbb{A}_{Q_1^0}, \mathbb{K}(S_1))$ and $(\mathbb{A}_{Q_2^0}, \mathbb{K}(S_2))$ are all equivalent.*

Proof. Note that $Q^0 = Q_1^0 \cup Q_2^0$, where the base points 0 of Q_1 and Q_2 are identified. From $Q_1^0 \cdot A_2A_1 = \{0\} = Q_2^0 \cdot A_1A_2$, it follows that S_1, S_2 are the subsemigroups of S generated by A_1A_2 and A_2A_1, respectively.

Next observe that $E(S) = E(S_1) \cup E(S_2)$. Indeed, since $A_1^2 = 0 = A_2^2$ in S, each non-zero idempotent $e \in S$ is represented by an alternating word u in A_1 and A_2. From $0 \neq e = e^2 = u^2$, we conclude that the first and last letters of u cannot both be in A_i, for either $i = 1, 2$. Thus $e \in S_1 \cup S_2$.

Since $S_1S_2 = 0 = S_2S_1$, it is now immediate that if $0 \neq s \in S_1$ and $e, f \in E(S)$, then $esf = s$ implies $e, f \in S_1$. Thus $\mathbb{K}(S_1)$ is a full subcategory of S (that is, the inclusion functor is full.) To obtain that the inclusion is an equivalence of categories, we need to show that each $e \in E(S)$ is isomorphic, or equivalently $\mathcal{D}$-equivalent, to an idempotent of S_1. By the previous paragraph, we may assume that $e \in E(S_2) \setminus \{0\}$ and that $e = a_2va_1$ where $a_2 \in A_2$, $a_1 \in A_1$ and $v \in S_1$. Lemma 2.4 implies $f = (va_1a_2)^2 \in S_1$ is an idempotent with $e \mathrel{\mathcal{D}} f$, that is $e \cong f$ in $\mathbb{K}(S)$. This proves that the inclusion functor $F\colon \mathbb{K}(S_1) \longrightarrow \mathbb{K}(S)$ is an equivalence.

Next let $\eta_e\colon Q_1^0e \longrightarrow Q^0e$ be the inclusion for each $e \in E(S_1)$. Then it is immediate that $\eta\colon \mathbb{A}_{Q_1^0} \Rightarrow \mathbb{A}_{Q^0} \circ F$ is a natural transformation. Clearly, each η_e is injective. For surjectivity, we use that $Q_2^0e = \{0\}$ for $e \in E(S_1)$ and hence $Q^0e = Q_1^0e$. A symmetric argument for S_2 completes the proof. $\square$

12.2. **Invariance under symbol expansion.** The reader should review the definitions and notation from Subsection 3.7.

Remark 12.2. Using induction on the length of words, one verifies that

$$\mathcal{E}(A^*) = B^* \setminus \Big(\diamond B^* \cup B^*\alpha \cup \bigcup_{x \in A\setminus\{\diamond\}} B^*\alpha x B^* \cup \bigcup_{x \in A\setminus\{\alpha\}} B^* x \diamond B^*\Big).$$

Remark 12.2 justifies several simple and useful facts, like the following.

Lemma 12.3. *Let $v \in A^+$. For $x, y, u \in B^*$, if $x\mathcal{E}(v)y = \mathcal{E}(u)$ then $x, y \in \mathcal{E}(A^*)$ and $u = \mathcal{E}^{-1}(x)v\mathcal{E}^{-1}(y)$. Consequently, if $\mathcal{E}(v) \in L(\mathscr{X}')$ then $v \in L(\mathscr{X})$.*

Proof. The first part of the lemma follows from Remark 12.2 and the fact that $\mathcal{E}$ is injective. If $\mathcal{E}(v) \in L(\mathscr{X}')$ then there is $u \in L(\mathscr{X})$ and $x, y \in B^*$ with $\mathcal{E}(u) = x\,\mathcal{E}(v)\,y$. From $u = \mathcal{E}^{-1}(x)\,v\,\mathcal{E}^{-1}(y)$, we deduce $v \in L(\mathscr{X})$. $\square$

A direct consequence of Lemma 12.3 is the following analog.

Lemma 12.4. *Let $x \in A^{\mathbb{Z}}$. If $\mathcal{E}(x) \in \mathscr{X}'$ then $x \in \mathscr{X}$.*

Proof. If u is a finite block of x, then $\mathcal{E}(u) \in L(\mathscr{X}')$. By Lemma 12.3, we have $u \in L(\mathscr{X})$. Hence $x \in \mathscr{X}$. $\square$

In [45], a semigroup homomorphism $\theta\colon S \longrightarrow T$ is termed a *local isomorphism* if the following conditions are satisfied:[3]

(1) $\theta|_{eSf}$ is a bijection of eSf with $\theta(e)T\theta(f)$;
(2) if $e' \in E(\theta(S))$, then there is $e \in E(S)$ with $\theta(e) = e'$;
(3) for each $e \in E(T)$, there is $f \in E(\theta(S))$ with $e \mathrel{\mathcal{D}} f$.

Remark 12.5. It is immediate from the definition (cf. [45]) that if $\theta\colon S \longrightarrow T$ is a local isomorphism, then θ induces an equivalence $\Theta\colon \mathbb{K}(S) \longrightarrow \mathbb{K}(T)$ given by $\Theta(e) = \theta(e)$ on objects and $\Theta(e, s, f) = (\theta(e), \theta(s), \theta(f))$ on morphisms.

Proposition 12.6. *There is a well-defined homomorphism $\mathcal{E}'\colon S(\mathscr{X}) \longrightarrow S(\mathscr{X}')$ sending $[u]_{\mathscr{X}}$ to $[\mathcal{E}(u)]_{\mathscr{X}'}$ and 0 to 0. Moreover, $\mathcal{E}'$ is a local isomorphism.*

Proof. We begin by showing that $[u]_{\mathscr{X}} \subseteq [v]_{\mathscr{X}}$ if and only if $[\mathcal{E}(u)]_{\mathscr{X}'} \subseteq [\mathcal{E}(v)]_{\mathscr{X}'}$ for $u, v \in A^+$. Suppose that $[u]_{\mathscr{X}} \subseteq [v]_{\mathscr{X}}$ and let x and y be words such that $x\,\mathcal{E}(u)\,y$ belongs to $L(\mathscr{X}')$. There are words x' and y' such that $x'x\,\mathcal{E}(u)\,yy'$ belongs to $L(\mathscr{X}') \cap \mathcal{E}(A^+)$. By Lemma 12.3, we have $\mathcal{E}^{-1}(x'x)\,u\,\mathcal{E}^{-1}(y'y) \in L(\mathscr{X})$. Since $[u]_{\mathscr{X}} \subseteq [v]_{\mathscr{X}}$, it follows that the word $z = \mathcal{E}^{-1}(x'x)\,v\,\mathcal{E}^{-1}(y'y)$ also belongs to $L(\mathscr{X})$. Hence $x\,\mathcal{E}(v)\,y$ belongs to $L(\mathscr{X}')$, since it is a factor of $\mathcal{E}(z)$. Therefore $[\mathcal{E}(u)]_{\mathscr{X}'} \subseteq [\mathcal{E}(v)]_{\mathscr{X}'}$. Let $z \in A^+ \setminus L(\mathscr{X})$. Then $\mathcal{E}(z) \notin L(\mathscr{X}')$, again by Lemma 12.3. Therefore, we have $\mathcal{E}'([z]_{\mathscr{X}}) = 0 = \mathcal{E}'(0)$. This proves $\mathcal{E}'$ is a well-defined homomorphism.

On the other hand, if $[\mathcal{E}(u)]_{\mathscr{X}'} \subseteq [\mathcal{E}(v)]_{\mathscr{X}'}$ then, for every $x, y \in A^*$, we have the following chain of implications, where the last one uses Lemma 12.3:

$$xuy \in L(\mathscr{X}) \Rightarrow \mathcal{E}(xuy) \in L(\mathscr{X}') \Rightarrow \mathcal{E}(xvy) \in L(\mathscr{X}') \Rightarrow xvy \in L(\mathscr{X}).$$

This shows that $[u]_{\mathscr{X}} \subseteq [v]_{\mathscr{X}}$. It follows that $\mathcal{E}'$ is injective. In particular, $\mathcal{E}'$ satisfies the second condition in the definition of a local isomorphism.

Suppose that $f = [w]_{\mathscr{X}'}$ is an idempotent of $S(\mathscr{X}')$ with $f \notin \mathcal{E}'(S(\mathscr{X}))$. Then $w \notin \mathcal{E}(A^+)$ on the one hand, and $w \in L(\mathscr{X}')$, on the other hand (the latter because $f \neq 0$.) As $w \in L(\mathscr{X}')$, there is $v \in L(\mathscr{X})$ such that

[3] Actually, Lawson only defines the notion for semigroups with local units.

$\mathcal{E}(v) = pwq$ for some p, q. Let u be a (possibly empty) word of maximal length such that $\mathcal{E}(u)$ is a factor of w, and let a, b be words with $w = a\,\mathcal{E}(u)\,b$. Since $\mathcal{E}(v) = pa\,\mathcal{E}(u)\,bq$, it follows that $pa, bq \in \mathcal{E}(A^+)$ by Lemma 12.3. By the maximality of u, we have $a, b \in \{1, \alpha, \diamond\}$. Note also that $\{a, b\} \neq \{1\}$, because $w \notin \mathcal{E}(A^+)$. Since $[w]_{\mathscr{X}'}$ is idempotent, the word $w^2 = a\,\mathcal{E}(u)\,ba\,\mathcal{E}(u)\,b$ belongs to $L(\mathscr{X}')$, thus $ra\,\mathcal{E}(u)\,ba\,\mathcal{E}(u)\,bs \in \mathcal{E}(A^+)$ for some r, s. Then ba belongs to $\mathcal{E}(A^+)$ by Remark 12.2. The only possibility is $ba = \alpha\diamond$, thus $w = \diamond\,\mathcal{E}(u)\,\alpha$. Since $\mathcal{E}(\alpha u) = \alpha \diamond \mathcal{E}(u)$, it follows from Lemma 2.4 that $[\mathcal{E}(\alpha u)^2]_{\mathscr{X}'}$ is an idempotent in the image of $\mathcal{E}'$ which is $\mathcal{D}$-equivalent to e.

It remains to show that $\mathcal{E}'(eS(\mathscr{X})f) = \mathcal{E}'(e)S(\mathscr{X}')\mathcal{E}'(f)$, whenever $e, f \in E(S(\mathscr{X}))$. Let $u, v \in A^+$ be such that $e = [u]_{\mathscr{X}}$ and $f = [v]_{\mathscr{X}}$. We clearly have $0 \in \mathcal{E}'(eS(\mathscr{X})f)$. Take $w \in L(\mathscr{X}')$ such that $[w]_{\mathscr{X}'} \in \mathcal{E}'(e)S(\mathscr{X}')\mathcal{E}'(f)$. Since $[w]_{\mathscr{X}'} = [\mathcal{E}(u)\,w\,\mathcal{E}(v)]_{\mathscr{X}'}$, there are words p, q such that $p\,\mathcal{E}(u)\,w\,\mathcal{E}(v)\,q$ belongs to $L(\mathscr{X}') \cap \operatorname{Im}\mathcal{E}$. From Remark 12.2 it follows that $w = \mathcal{E}(w')$ for some $w' \in A^+$. Moreover, $w' \in L(\mathscr{X})$ by Lemma 12.3. Clearly, $[uw'v]_{\mathscr{X}} \in eS(\mathscr{X})f$. Furthermore, $\mathcal{E}'[uw'v]_{\mathscr{X}} = \mathcal{E}'(e)\,[w]_{\mathscr{X}'}\,\mathcal{E}'(f) = [w]_{\mathscr{X}'}$, completing the proof. $\square$

Consider the mapping $F_{\mathcal{E}} \colon \mathbb{K}(\mathscr{X}) \longrightarrow \mathbb{K}(\mathscr{X}')$ defined as follows:

(1) $F_{\mathcal{E}}(e) = \mathcal{E}'(e)$ if e is an object of $\mathbb{K}(\mathscr{X})$;
(2) $F_{\mathcal{E}}(e, s, f) = \big(\mathcal{E}'(e), \mathcal{E}'(s), \mathcal{E}'(f)\big)$ if (e, s, f) is morphism of $\mathbb{K}(\mathscr{X})$.

In view of Remark 12.5 and Proposition 12.6, we know that $F_{\mathcal{E}}$ is an equivalence. This completes the proof of Theorem 4.2.

We proceed with the proof of Theorem 4.3, by steps, in order to produce an isomorphism $\mathbb{A}_{\mathscr{X}} \Rightarrow \mathbb{A}_{\mathscr{X}'} \circ F_{\mathcal{E}}$.

Lemma 12.7. *The inclusion $C_{\mathscr{X}}(x) \subseteq C_{\mathscr{X}}(y)$ holds if and only if the inclusion $C_{\mathscr{X}'}(\mathcal{E}(x)) \subseteq C_{\mathscr{X}'}(\mathcal{E}(y))$ holds, whenever $x, y \in A^{\mathbb{Z}^-}$.*

Proof. Suppose that $C_{\mathscr{X}'}(\mathcal{E}(x)) \subseteq C_{\mathscr{X}'}(\mathcal{E}(y))$. Let $z \in C_{\mathscr{X}}(x)$, that is $x.z \in \mathscr{X}$. Then $\mathcal{E}(x).\mathcal{E}(z) = \mathcal{E}(x.z) \in \mathscr{X}'$. Hence $\mathcal{E}(y).\mathcal{E}(z) = \mathcal{E}(y.z) \in \mathscr{X}'$, by hypothesis. By Lemma 12.4, we have $y.z \in \mathscr{X}$, showing $C_{\mathscr{X}}(x) \subseteq C_{\mathscr{X}}(y)$.

Conversely, assume $C_{\mathscr{X}}(x) \subseteq C_{\mathscr{X}}(y)$. Let $z \in C_{\mathscr{X}'}(\mathcal{E}(x))$, meaning $\mathcal{E}(x).z \in \mathscr{X}'$. By Remark 12.2, we have $z = \mathcal{E}(t)$ for some unique $t \in A^{\mathbb{N}}$. Moreover, $x.t \in \mathscr{X}$ by Lemma 12.4. Since $C_{\mathscr{X}}(x) \subseteq C_{\mathscr{X}}(y)$, we obtain $y.t \in \mathscr{X}$, whence $\mathcal{E}(y).z \in \mathscr{X}'$. Therefore, $C_{\mathscr{X}'}(\mathcal{E}(x)) \subseteq C_{\mathscr{X}'}(\mathcal{E}(y))$. $\square$

Consider the map $h \colon Q(\mathscr{X}) \longrightarrow Q(\mathscr{X}')$ with $h(C_{\mathscr{X}}(x)) = C_{\mathscr{X}'}(\mathcal{E}(x))$, for every $x \in A^{\mathbb{Z}^-}$. By Lemma 12.7, this is a well-defined injective function.

Lemma 12.8. *We have $h(q \cdot s) = h(q) \cdot \mathcal{E}'(s)$, for every $q \in Q(\mathscr{X})$ and $s \in S(\mathscr{X})$.*

Proof. Take $u \in A^+$ with $s = [u]_{\mathscr{X}}$, and $x \in A^{\mathbb{Z}^-}$ with $q = C_{\mathscr{X}}(x)$. Then $q \cdot s = C_{\mathscr{X}}(x)$ and $h(q \cdot s) = C_{\mathscr{X}'}(\mathcal{E}(xu)) = C_{\mathscr{X}'}(\mathcal{E}(x)) \cdot [\mathcal{E}(u)]_{\mathscr{X}'}$. Since $\mathcal{E}'(s) = [\mathcal{E}(u)]_{\mathscr{X}'}$, this concludes the proof. $\square$

Lemma 12.9. *The equality $h(Q(\mathscr{X})\cdot s) = Q(\mathscr{X}')\cdot\mathcal{E}'(s)$ holds for every $s\in S(\mathscr{X})$.*

Proof. By Lemma 12.8, we have $h(Q(\mathscr{X})\cdot s)\subseteq Q(\mathscr{X}')\cdot\mathcal{E}'(s)$. Conversely, let $q\in Q(\mathscr{X}')\cdot\mathcal{E}'(s)$. We want to show that $q\in h(Q(\mathscr{X})\cdot s)$. Since $\emptyset=\emptyset\cdot 0$, by Lemma 12.8 we have $h(\emptyset)=h(\emptyset)\cdot 0=\emptyset$. Therefore, we may suppose $q\neq\emptyset$. Take $u\in A^+$ such that $s=[u]_{\mathscr{X}}$. Then, there is $y\in B^{\mathbb{Z}^-}$ such that $q=C_{\mathscr{X}'}(y\,\mathcal{E}(u))$. The assumption $q\neq 0$ means that $y\,\mathcal{E}(u).z\in\mathscr{X}'$ for some $z\in B^{\mathbb{N}}$, and so by Remark 12.2 there is $\tilde{y}\in B^{\mathbb{Z}^-}$ such that $y=\mathcal{E}(\tilde{y})$. Then $h(C_{\mathscr{X}}(\tilde{y}u))=q$. Since $C_{\mathscr{X}}(\tilde{y}u)\in Q(\mathscr{X})\cdot s$, this concludes the proof. □

We are now ready to exhibit an isomorphism $\mathbb{A}_{\mathscr{X}}\Rightarrow\mathbb{A}_{\mathscr{X}'}\circ F_{\mathcal{E}}$.

Proposition 12.10. *For each idempotent $e\in S(\mathscr{X})$, let η_e be the function $Q(\mathscr{X})e\longrightarrow Q(\mathscr{X}')\,\mathcal{E}'(e)$ such that $\eta_e(r)=h(r)$ for every $r\in Q(\mathscr{X})e$. Let $\eta=(\eta_e)_{e\in E(S(\mathscr{X}))}$. Then η is an isomorphism $\mathbb{A}_{\mathscr{X}}\Rightarrow\mathbb{A}_{\mathscr{X}'}\circ F_{\mathcal{E}}$.*

Proof. By Lemma 12.9, the range of η_e is correctly defined, and η_e is bijective (as h is injective.) On the other hand, by Lemma 12.8, the family η is a natural transformation from $\mathbb{A}_{\mathscr{X}}$ to $\mathbb{A}_{\mathscr{X}'}\circ F_{\mathcal{E}}$. □

Conclusion of the proof of Theorems 4.2 and 4.3. As remarked in Subsection 12.1, the conjugacy invariance part of Theorem 4.3 follows immediately from Theorem 3.9 and Lemma 12.1. The symbol expansion part is contained in Proposition 12.10. Since flow equivalence is generated by conjugacy and symbol expansion, we are done. □

12.3. **Proof of Theorem 4.5.** We are ready to show Theorem 4.5

Conclusion of the proof of Theorem 4.5. Let $\mathscr{X}$ and $\mathscr{Y}$ be flow equivalent synchronizing shifts. By Theorem 4.3, there is an equivalence $F\colon\mathbb{K}(\mathscr{X})\longrightarrow\mathbb{K}(\mathscr{Y})$ for which there is an isomorphism $\eta\colon\mathbb{A}_{\mathscr{X}}\longrightarrow\mathbb{A}_{\mathscr{Y}}\circ F$.

Since $Q_{\mathfrak{F}}(\mathscr{X})$ is an $S(\mathscr{X})$-invariant subset of $Q(\mathscr{X})$, it follows that $\mathbb{A}^{\mathfrak{F}}_{\mathscr{X}}$ is a subfunctor of $\mathbb{A}_{\mathscr{X}}$. Similarly, $\mathbb{A}^{\mathfrak{F}}_{\mathscr{Y}}$ is a subfunctor of $\mathbb{A}_{\mathscr{Y}}$. Hence, the result will follow immediately as along as $\eta_e(Q_{\mathfrak{F}}(\mathscr{X})e)=Q_{\mathfrak{F}}(\mathscr{Y})$ for all $e\in E(S(\mathscr{X}))$. But this is the content of Remark 5.3. □

13. The proof of Theorem 4.6

The proof of Theorem 4.6 relies on the next two propositions.

Proposition 13.1. *Let S and T be finite semigroups with zero having local units. If S and T are Morita equivalent, then there are shifts $\mathscr{X}_S$ and $\mathscr{X}_T$, respectively induced by S and T, such that $\mathscr{X}_S$ and $\mathscr{X}_T$ are conjugate.*

Proof. Since S and T are Morita equivalent finite semigroups with local units, it follows by the results of Lawson [45] that there is a semigroup R containing S and T (more precisely, disjoint isomorphic copies of S and T) such that $SRS=S$, $RSR=R$, $TRT=T$ and $RTR=R$. Moreover, Lawson's proof shows that R can be taken to be finite. Note that since S,T have local units, it follows that $T = TT$ and $S = SS$, and so we have

$$S = SRT \cdot TRS, \quad T = TRS \cdot SRT. \tag{13.1}$$

This suggests to take the finite sets $C = S\times R\times T$, $D = T\times R\times S$, $A = C\times D$ and $B = D \times C$. Thanks to (13.1), we may consider the homomorphisms $\varphi\colon A^+ \longrightarrow S$ and $\psi\colon B^+ \longrightarrow T$ defined by

$$\varphi\big((s_1,r_1,t_1),(t_2,r_2,s_2)\big) = s_1r_1t_1t_2r_2s_2,$$
$$\psi\big((t_2,r_2,s_2),(s_1,r_1,t_1)\big) = t_2r_2s_2s_1r_1t_1,$$

for every $s_i \in S$, $r_i \in R$, $t_i \in T$, $i \in \{1,2\}$. Also by (13.1), we have $S = \varphi(A)$ and $T = \psi(B)$, thus φ and ψ are onto. Hence, we may consider the shifts $\mathscr{X}_S = \mathscr{X}_\varphi \subseteq A^{\mathbb{Z}}$ and $\mathscr{X}_T = \mathscr{X}_\psi \subseteq B^{\mathbb{Z}}$, respectively induced by φ and ψ. The proof is completed once we show that $\mathscr{X}_S$ and $\mathscr{X}_T$ are conjugate.

Consider the map $f\colon A^{\mathbb{Z}} \longrightarrow B^{\mathbb{Z}}$ given by $f((c_i,d_i)_{i\in\mathbb{Z}}) = (d_i,c_{i+1})_{i\in\mathbb{Z}}$. One sees straightforwardly that f is a conjugacy (it is actually a *bipartite code*, an important special class of conjugacies introduced by Nasu [51]). Its inverse is $f^{-1}((d_i,c_i)_{i\in\mathbb{Z}}) = (c_{i-1},d_i)_{i\in\mathbb{Z}}$. We claim that $f(\mathscr{X}_S) = \mathscr{X}_T$. Denote by 0_S the zero of S, and by 0_T the zero of T. Let

$$x = \dots(c_{-2},d_{-2})(c_{-1},d_{-1}).(c_0,d_0)(c_1,d_1)(c_2,d_2)\dots$$

be an element of $A^{\mathbb{Z}}$. Suppose we have $f(x) \notin \mathscr{X}_T$. Then, there are $i,j \in \mathbb{Z}$, with $i \leq j$, such that $\psi\big((d_i,c_{i+1})(d_{i+1},c_{i+2})\cdots(d_j,c_{j+1})\big) = 0_T$. Take $s = \varphi\big((c_i,d_i)(c_{i+1},d_{i+1})\cdots(c_j,d_j)(c_{j+1},d_{j+1})\big)$. To show that $x \notin \mathscr{X}_S$ it suffices to show $s = 0_S$. Let $c_i = (s_1,r_1,t_1)$, $d_j = (t_2,r_2,s_2)$. Then, we have in R the factorization $s = s_1r_1t_10_Tt_2r_2s_2$. Also, we have

$$0_S = s0_Ss = s_1r_1t_10_Tt_2r_2s_2 \cdot 0_S \cdot s_1r_1t_10_Tt_2r_2s_2 = s_1r_1t_10_Tz0_Tt_2r_2s_2, \tag{13.2}$$

where $z = t_2r_2s_20_Ss_1r_1t_1$. By (13.1), we have $z \in T$, thus $0_Tz0_T = 0_T$, and so from (13.2) we get $0_S = s_1r_1t_10_Tt_2r_2s_2 = s$. This shows $x \notin \mathscr{X}_S$, establishing $f(\mathscr{X}_S) \subseteq \mathscr{X}_T$. Similarly, one shows $f^{-1}(\mathscr{X}_T) \subseteq \mathscr{X}_S$. Therefore, we have $f(\mathscr{X}_S) = \mathscr{X}_T$, whence $\mathscr{X}_S$ and $\mathscr{X}_T$ are conjugate. □

Proposition 13.2. *Let $\mathscr{X}$ be a sofic shift. Then there is a sofic shift $\mathscr{Z}$ which is flow equivalent to $\mathscr{X}$ and such that $S(\mathscr{Z})$ has local units.*

Proof. Let $\delta\colon A^+ \longrightarrow S(\mathscr{X})$ be the syntactic homomorphism. Say that $u \in A^+$ is *δ-idempotent* if $\delta(u)$ is idempotent. A δ-idempotent word is *minimal* if it has not proper factors which are δ-idempotent. Note that every δ-idempotent word has some minimal δ-idempotent factor.

Say also that $u \in A^+$ is *δ-special* if it has a proper prefix and proper suffix which are minimal δ-idempotents. A δ-special word u is *minimal* if every proper factor of u is not δ-special. In a more informal and intuitive manner, one can say that a minimal δ-special word represents two consecutive occurrences of minimal δ-idempotents words in an element of $A^{\mathbb{Z}}$.

Denote by W the set of minimal δ-special words of A^+. Since $S(\mathscr{X})$ is finite, there is an integer $N \geq 1$ such that every element of A^+ with length N has a δ-idempotent factor (cf. [57, Theorem 1.11].) Therefore, every word of A^+ with length at least $2N$ has at least one factor which is minimal δ-special, and so the elements of W have length at most $2N$. In particular, W is finite.

For $u \in W$, let $\tau(u) = (p, u, q)$ be the triple such that p and q are respectively the unique minimal δ-idempotent prefix and the unique minimal δ-idempotent suffix of u. Let V be the set $\{\tau(u) \mid u \in W\}$, which is in bijection with W. For future reference, we extend the bijection τ as follows. Let u be a δ-special word, not necessarily minimal. We define $\bar{\tau}(u) \in V^+$ recursively on the number of occurrences of minimal δ-idempotent words, as follows:

(1) if v is minimal δ-special, then $\bar{\tau}(u) = \tau(u)$ and the recursion stops;
(2) if u is not minimal δ-special, then u admits a factorization $u = vqw$ in A^+, such that vq is minimal δ-special and q is minimal δ-idempotent; in particular, qw is δ-special with a number of occurrences of minimal δ-idempotent words smaller than that of u; then we let $\bar{\tau}(u) = \tau(uq)\bar{\tau}(qw)$.

For $u \in W$, let $\tau(u) = (p, u, q)$. We have a factorization $u = zq$. Define $m(u) = |z| - 1$. Consider a set $\Lambda(u) = \{\diamond_{u,1}, \diamond_{u,2}, \ldots, \diamond_{u,m(u)}\}$ of $m(u)$ elements not in V. We use the convention that $\diamond_{u,0} = \tau(u)$. Assume moreover that if u, v are distinct elements of W, then $\Lambda(u) \cap \Lambda(v) = \emptyset$. Denote by B the alphabet $V \cup \bigl(\bigcup_{u \in W} \Lambda(u)\bigr)$.

Take $x \in A^{\mathbb{Z}}$. Denote by $P(x)$ the set of occurrences in x of elements of W, which is precisely the set of occurrences in x of minimal δ-idempotents. If $i \in P(x)$, denote by $w_x(i)$ the element of W occurring in x at position i, where we say that a word w *occurs* at i in x if $x_{[i,j]} = w$, for some j. For each $i \in \mathbb{Z}$, let $\kappa_x(i) = \max\{j \in P(x) \mid j \leq i\}$ and consider the difference $d_x(i) = i - \kappa_x(i) \geq 0$. We may then define $w_x(i)$ for all $i \in \mathbb{Z}$ as being $w_x(i) = w_x(\kappa_x(i))$. That is, if i, j are consecutive occurrences of elements of $P(x)$, then $w_x(k) = w_x(i)$ for every k such that $i \leq k < j$. For each $i \in \mathbb{Z}$, let $\tau(w_x(i)) = (p_x(i), w_x(i), q_x(i))$. Note that if i, j are consecutive elements of $P(x)$ then $q_x(i) = p_x(j)$.

By the maximality of $\kappa_x(i)$, the word $q_x(i)$ cannot occur at a position belonging to $\{j \mid \kappa_x(i) < j \leq i\}$, and so $d_x(i) \leq m(w_x(i))$, with equality $d_x(i) = m(w_x(i))$ if and only if $i + 1 \in P(x)$. Consider the mapping $f \colon A^{\mathbb{Z}} \longrightarrow B^{\mathbb{Z}}$ given by $f(x) = (\diamond_{w_x(i), d_x(i)})_{i \in \mathbb{Z}}$, $x \in A^{\mathbb{Z}}$. More intuitively, f is characterized by the following property: if $i, i + n$ are two consecutive elements of $P(x)$ (and so $m(w_x(i)) = n - 1$), then the sequence

$$(p_x(i), w_x(i), q_x(i))\, \diamond_{w_x(i),1} \cdots \diamond_{w_x(i),n-1}\, (q_x(i), w_x(i + n), q_x(i + n))$$

is an word of B^+ at position i in $f(x)$.

For each $i \in \mathbb{Z}$, if $d_x(i) < m(w_x(i))$ then $\kappa_{\sigma(x)}(i) = \kappa_x(i) - 1$ and if $d_x(i) = m(w_x(i))$ then $\kappa_{\sigma(x)}(i) = i$. In both cases, the word $w_{\sigma(x)}(i)$ is equal to $w_x(i+1)$. This amounts to say that f commutes with the shift mapping. Observe also that $f(x)_i$ is determined by $x_{[i-N,i+N]}$, and so f is continuous, whence a morphism of shifts.

Let $x, y \in A^{\mathbb{Z}}$, and let $i \in \mathbb{Z}$. We have $f(x)_i = \diamond_{u,d_x(i)}$ and $f(y)_i = \diamond_{v,d_x(i)}$, for some $u, v \in W$. Moreover, x_i is the letter at position $d_x(i)+1$ in u, and y_i is the letter at position $d_y(i)+1$ in v. Therefore, we have $x_i = y_i$ when $\diamond_{u,d_x(i)} = \diamond_{v,d_y(i)}$, as the latter implies $u = v$ and $d_x(i) = d_y(i)$. Hence, $x_i \neq y_i$ implies $f(x)_i \neq f(y)_i$, showing that f is injective. We conclude that f induces a conjugacy between $\mathscr{X}$ and the shift $\mathscr{Y} = f(\mathscr{X})$.

Take $u \in W$ and $i \in \{1, \dots, m(u)\}$. In an element of $\mathscr{Y}$, an occurrence of $\diamond_{u,i}$ is always preceeded by an occurrence of $\diamond_{u,i-1}$. Hence, starting in $\mathscr{Y}$, we may perform a sequence C of symbol contractions $\diamond_{u,i-1}\diamond_{u,i} \mapsto \diamond_{u,i-1}$, with i running $\{1, \dots, m(u)\}$ and u running W, to obtain a subshift $\mathscr{Z}$ of $V^{\mathbb{Z}}$ which is flow equivalent to $\mathscr{X}$. We denote by f^C the mapping $\mathscr{X} \longrightarrow \mathscr{Z}$ assigning to each $x \in \mathscr{X}$ the element of $\mathscr{Z}$ obtained from $f(x)$ by applying the sequence C of symbol contractions.

Let $\mathscr{X}_0 = \{x \in \mathscr{X} \mid 0 \in P(x)\}$. Then every element of $\mathscr{Z}$ is in the orbit of an element of $f^C(\mathscr{X}_0)$. Let $\nu = (p_0, p_0t_0, p_1)$ be an arbitrary element of V. Suppose there is $x \in \mathscr{X}_0$ such that $f^C(x)$ equals

$$\dots (p_{-2}, z_{-2}p_{-1}, p_{-1})(p_{-1}, z_{-1}p_0, p_0).(p_0, p_0t_0, p_1)(p_1, p_1t_1, p_2)\dots. \tag{13.3}$$

The sequence x is completely determined by (13.3). More precisely, x equals

$$\dots z_{-2}z_{-1}.p_0t_0t_1t_2\dots. \tag{13.4}$$

Since $\delta(p_0)$ is idempotent, it follows that the sequence

$$\dots z_{-2}z_{-1}.p_0p_0t_0t_1t_2\dots \tag{13.5}$$

also belongs to $\mathscr{X}_0$. Its image under f^C is

$$\dots (p_{-2}, z_{-2}p_{-1}, p_{-1})(p_{-1}, z_{-1}p_0, p_0).\bar{\tau}(p_0p_0)(p_0, p_0t_0, p_1)(p_1, p_1t_1, p_2)\dots. \tag{13.6}$$

Comparing (13.3) and (13.6), we conclude that the context $[\nu]_{\mathscr{Z}}$ is contained in the context $[\bar{\tau}(p_0p_0)\nu]_{\mathscr{Z}}$. Let us verify that the reverse inclusion also holds. If the sequence (13.6) belongs to $\mathscr{Z}$, then the sequence (13.5) belongs to $\mathscr{X}_0$, and therefore so does (13.4) by the hypothesis that $\delta(p_0)$ is idempotent. We then deduce that the sequence (13.3) belongs to $\mathscr{Z}$, showing the desired inclusion of contexts. Hence, we have $[\nu]_{\mathscr{Z}} = [\bar{\tau}(p_0p_0)\nu]_{\mathscr{Z}}$, thus $[\nu]_{\mathscr{Z}} = s^n \cdot [\nu]_{\mathscr{Z}}$, for every $n \geq 1$, where $s = [\bar{\tau}(p_0p_0)]_{\mathscr{Z}}$. As $S(\mathscr{Z})$ is finite, there is $m \geq 1$ such that $s^m = e$ is idempotent (cf. [57, Proposition 1.6]), thus $[\nu]_{\mathscr{Z}} = e \cdot [\nu]_{\mathscr{Z}}$. Similarly, one can show that $[\nu]_{\mathscr{Z}} = [\nu\bar{\tau}(p_1p_1)]_{\mathscr{Z}}$ and that therefore we have $[\nu]_{\mathscr{Z}} = [\nu]_{\mathscr{Z}} \cdot e'$ for some idempotent e'. Therefore, $[\nu]_{\mathscr{Z}}$ has local units. As the elements of the form $[\nu]_{\mathscr{Z}}$, $\nu \in V$, generate $S(\mathscr{Z})$, we conclude that $S(\mathscr{Z})$ has local units. $\square$

We are now ready to prove Theorem 4.6.

Proof of Theorem 4.6. By Proposition 13.2, there are sofic shifts $\widetilde{\mathscr{X}}$ and $\widetilde{\mathscr{Y}}$, respectively flow equivalent to $\mathscr{X}$ and $\mathscr{Y}$, such that both $S(\widetilde{\mathscr{X}})$ and $S(\widetilde{\mathscr{Y}})$ have local units. Since ϑ is an invariant of flow equivalence, we have $\mathscr{X}\ \vartheta\ \widetilde{\mathscr{X}}$ and $\mathscr{Y}\ \vartheta\ \widetilde{\mathscr{Y}}$. Since the equivalence class of the Karoubi envelope is a flow invariant (Theorem 4.2), we know that $\mathbb{K}(\mathscr{X})$ is equivalent to $\mathbb{K}(\widetilde{\mathscr{X}})$ and $\mathbb{K}(\mathscr{Y})$ is equivalent to $\mathbb{K}(\widetilde{\mathscr{Y}})$. As $\mathbb{K}(\mathscr{X})$ is by hypothesis equivalent to $\mathbb{K}(\mathscr{Y})$, we conclude that $\mathbb{K}(\widetilde{\mathscr{X}})$ is equivalent to $\mathbb{K}(\widetilde{\mathscr{Y}})$, that is, $S(\widetilde{\mathscr{X}})$ and $S(\widetilde{\mathscr{Y}})$ are Morita equivalent. By Proposition 13.1, we know there are conjugate shifts $\mathscr{X}^\sharp$ and $\mathscr{Y}^\sharp$ respectively induced by $S(\widetilde{\mathscr{X}})$ and $S(\widetilde{\mathscr{Y}})$. Again by the definition of flow invariant of sofic shifts, we have $\mathscr{X}^\sharp\ \vartheta\ \mathscr{Y}^\sharp$. But by Lemma 3.2, we have $S(\widetilde{\mathscr{X}}) = S(\mathscr{X}^\sharp)$ and $S(\widetilde{\mathscr{Y}}) = S(\mathscr{Y}^\sharp)$. So from the definition of syntactic invariant of sofic shifts we have that $\mathscr{X}\ \vartheta\ \mathscr{X}^\sharp$ and $\mathscr{Y}\ \vartheta\ \mathscr{Y}^\sharp$. Thus $\mathscr{X}\ \vartheta\ \mathscr{Y}$, as required. □

Appendix A. Symbol expansion and subsynchronizing subshifts

Now that we have the tools developed in Subsection 12.2, we are able to show the proposition needed to conclude the proof of Theorem 10.7.

Proposition A.1. *Let $\mathscr{X}$ be a sofic shift, and let e be a magic idempotent of $S(\mathscr{X})$. Consider a symbol expansion $\mathscr{X}'$ of $\mathscr{X}$, defined by a symbol expansion homomorphism $\mathcal{E}$. If e is a magic idempotent for $\mathscr{X}$, then the subsynchronizing subshift $S(F_{\mathcal{E}}(e))$ of $\mathscr{X}'$ is the symbol expansion of $S(e)$ defined by $\mathcal{E}$.*

Proof. Let $u \in L(\mathscr{X})$ be such that $e = [u]_{\mathscr{X}}$.

Consider an element of $L(S(e)')$ of the form $\mathcal{E}(v)$, with $v \in L(S(e))$. Then $uwv \in L(\mathscr{X})$ for w, thus $\mathcal{E}(uwv) \in L(\mathscr{X}')$. Since $F_{\mathcal{E}}(e) = [\mathcal{E}(u)]_{\mathscr{X}'}$, this shows that $\mathcal{E}(v)$ is a finite block of $S(F_{\mathcal{E}}(e))$. Every finite block of $S(e)'$ is a factor of a word such as $\mathcal{E}(v)$. Therefore, we proved that $S(e)' \subseteq S(F_{\mathcal{E}}(e))$.

Conversely, let v be a finite block of $S(F_{\mathcal{E}}(e))$. Then $\mathcal{E}(u)wvw'\mathcal{E}(u') \in L(\mathscr{X}')$ for some w, w', u', with u' a non-empty word over the alphabet of $\mathscr{X}$. By Remark 12.2, $\mathcal{E}(u)wvw'\mathcal{E}(u')$ and wvw' are in $\mathrm{Im}\mathcal{E}$, thus $u\,\mathcal{E}^{-1}(wvw')u' \in L(\mathscr{X})$. This establishes $\mathcal{E}^{-1}(wvw') \in S(e)$. Hence, wvw' belongs to $L(S(e)')$, and hence so does its factor v. This shows that $S(F_{\mathcal{E}}(e)) \subseteq S(e)'$. □

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